Bachelor of Science

(B.Sc.)

CALCULUS OF SINGLE VARIABLES

Semester-I

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Course overview

Using the idea of definite integral developed in previous semester, the natural logarithm function is defined and its properties are examined. This allows us to define its inverse function namely the natural exponential function and also the general exponential function. Exponential functions model a wide variety of phenomenon of interest in science, engineering, mathematics and economics. They arise naturally when we model the growth of a biological population, the spread of a disease, the radioactive decay of atoms, and the study of heat transfer problems and so on. We also consider certain combinations of exponential functions namely hyperbolic functions that also arise very frequently in applications such as the study of shapes of cables hanging under their own weight. After this, the students are introduced to the idea of improper integrals, their convergence and evaluation. This enables to study a related notion of convergence of a series, which is practically done by applying several different tests such as integral test, comparison test and so on. As a special case, a study on power series- their region of convergence, differentiation and integration etc.,- is also done. A detailed study of plane and space curves is then taken up. The students get the idea of parametrization of curves, they learn how to calculate the arc length, curvature etc. using parametrization and also the area of surface of revolution of a parametrized plane curve. Students are introduced into other coordinate systems which often simplify the equation of curves and surfaces and the relationship between various coordinate systems are also taught. This enables them to directly calculate the arc length and surface areas of revolution of a curve whose equation is in polar form. At the end of the course, the students will be able to handle vectors in dealing with the problems involving geometry of lines, curves, planes and surfaces in space and have acquired the ability to sketch curves in plane and space given in vector valued form.

Module 1

The Transcendental functions

1.1 The natural logarithmic function

Definition 1. *The natural logarithmic function, denoted by ln, is the function defined by*

$$lnx = \int_1^x \frac{1}{t} dt$$

for all $x > 0$.

Definition 2. ***(Derivative of** lnx) Using Fundamental theorem of Calculus, we get that*

$$\frac{d}{dx} lnx = \frac{d}{dx} \int_1^x \frac{1}{t} dt = \frac{1}{x} \quad x > 0$$

Laws of logarithms : Let x and y be positive numbers and let r

be a rational. Then

a. $ln x = 0$

b. $ln xy = ln x + ln y$

c. $ln \frac{x}{y} = ln x - ln y$

d. $ln x^r = r ln x$

Example : Expand the expression $ln\frac{x^2+1}{\sqrt{x}}$.

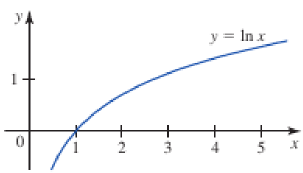
Solution :

$$\begin{aligned}\frac{x^2+1}{\sqrt{x}} &= \frac{x^2+1}{x^{1/2}}\\ &= ln(x^2+1) - ln(x^{1/2})\\ &= ln(x^2+1) - \frac{1}{2}ln x\end{aligned}$$

Graph of the natural logarithmic function : $f(x) = ln x$ has the following properties:

1. The domain of f is $(0, \infty)$, by definition.
2. f is continuous on $(0, \infty)$, since it is differentiable there.
3. f is increasing on $(0, \infty)$, since $f'(x) = \frac{1}{x} > 0$ on $(0, \infty)$.
4. The graph of f is concave downward on $(0, \infty)$ since $f''(x) = -\frac{1}{x^2} < 0$ on $(0, \infty)$.

Using these properties and the results $\lim_{x\to 0^+} ln x = -\infty$ and $\lim_{x\to\ infty} ln x = \infty$
we sketch the graph of $f(x) = ln x$, as shown below.



Theorem 1. *(Derivative of the Natural logarithmic function) Let u be a differential function of x. then*
a. $\frac{d}{dx}ln|x| = \frac{1}{x}\ x \neq 0$
b. $\frac{d}{dx}ln|u| = \frac{1}{u}.\frac{du}{dx}\ u \neq 0$

Proof. a. If $x > 0$, then as we have have already seen earlier

$$\frac{d}{dx}ln|x| = \frac{d}{dx}lnx = \frac{1}{x}$$

If $x < 0$, then $|x| = -x$. So we have

$$\frac{d}{dx}ln|x| = \frac{d}{dx}ln(-x) = \frac{1}{x}$$

b. This follows from the Chain Rule. □

Example : Find the derivative of
a. $f(x) = ln(2x^2 + 1)$ b. $g(x) = x^2ln2x$
Solution :
a. $f'(x) = \frac{d}{dx}f(x) = ln(2x^2 + 1) = \frac{1}{2x^2+1}\frac{d}{dx}(2x^2 + 1) = \frac{4x}{2x^2+1}$
b.

$$\begin{aligned} g'(x) = \frac{d}{dx}(x^2ln2x) = x^2\frac{d}{dx}(ln2x) + (ln2x)\frac{d}{dx}(x^2) \\ = x^2(\frac{1}{2x})(2) + (ln2x)(2x) = x(1 + 2ln2x) \end{aligned}$$

Logarithmic Differentiation : We have seen how the laws of logarithms can help to simplify the work involved in differentiating logarithmic expressions.We now look at a procedure that takes advantage of these same laws to help us differentiate functions that at first blush do not necessarily involve logarithms. This method, called logarithmic differentiation, is especially useful for differentiating functions involving products, quotients, and/or powers that can be simplified by using logarithms.

Steps to finding $\frac{dy}{dx}$ by Logarithmic Differentiation : Suppose that we are given the equation $y = f(x)$. To compute $\frac{dy}{dx}$:
1. Take the logarithm of both sides of the equation, and use the laws of logarithms to simplify the resulting equation.
2. Differentiate implicitly with respect to x.
3. Solve the equation found in Step 2 for $\frac{dy}{dx}$.
4. Substitute for y.

Example : Find the derivative of $y = \frac{(2x-1)^3}{\sqrt{3x+1}}$.
Solution : We begin by taking the natural logarithm on both sides of the equation,

$$lny = ln\frac{(2x-1)^3}{\sqrt{3x+1}}$$

or

$$lny = 3ln(2x-1) - \frac{1}{2}ln(3x+1)$$

Now diferentiating with respect to x

$$\begin{aligned}\frac{1}{y}(y') &= \frac{3}{2x-1}(2) - \frac{1}{2(3x+1)}(3)\\ &= \frac{6}{2x-1} - \frac{3}{2(3x+1)}\\ &= \frac{6.2(3x+1) - 3(2x-1)}{2(2x-1)(3x+1)}\end{aligned}$$

This gives,

$$\begin{aligned} y' &= \frac{6.2(3x+1)-3(2x-1)}{2(2x-1)(3x+1)}.y \\ &= \frac{6.2(3x+1)-3(2x-1)}{2(2x-1)(3x+1)}.\frac{(2x-1)^3}{\sqrt{3x+1}} \\ &= \frac{15(2x+1)(2x-1)^2}{2(3x+1)^{3/2}} \end{aligned}$$

Integration involving Logarithmic functions : By reversing the rule

$$\frac{d}{dx}ln|u| = \frac{1}{u}\frac{du}{dx}$$

we obtain the following rule of integration.

Theorem 2. ***(Rule for integrating*** $\frac{1}{u}$***)*** *Let* $u = g(x)$ *, where* g *is differentiable, and suppose that* $g(x) \neq 0$*. Then*

$$\int \frac{1}{u}du = ln|u| + C$$

Example : Evaluate the following

a. $\int \frac{1}{2x+1}dx$

b. $\int tanx dx$

c. $\int sexx dx$

Solution : a. Let $u = 2x+1$, so that $du = 2dx$ or $dx = \frac{1}{2}du$. Making these substitutions, we get

$$\begin{aligned} \int \frac{1}{2x+1}dx = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\int ln|u| + C \\ = \frac{1}{2}ln|2x+1| + C \end{aligned}$$

b. Since $tanx = \frac{sinx}{cosx}$, we use the substitution $u = cosx$, so that $du = -sinxdx =$ or $sinxdx = -du$. This gives

$$\begin{aligned}\int tanxdx &= \int \frac{sinx}{cosx}dx = -\int \frac{1}{u}du \\ &= -ln|u| + C = -ln|cosx| + C \\ &= ln|secx| + C\end{aligned}$$

c. Multiplying both the numerator and denominator of the integrand by $secx + tanx$ gives

$$\int secxdx = \int secx\frac{secx + tanx}{secx + tanx}dx = \int \frac{sec^2x + secx.tanx}{secx + tanx}$$

Now, we use the substitution $u = secx + tanx$, so that we get $du = (secx.tanx + sec^2x)dx$. This gives,

$$\int secxdx = \int \frac{1}{u}du = ln|u| + C = ln|secx + tanx| + C$$

We can use the above technique to find the integral of other trigonometric functions, the results of which are summarized below.

Theorem 3. ***(Integrals of Trigonometric Functions)***
a. $\int tanudu = ln|secu| + C$
b. $\int cotudu = ln|sinu| + C$
c. $\int secudu = ln|secu + tanu| + C$
d. $\int cscudu = ln|cscu - cotu| + C$

Example : Find $\int xsexx^2dx$.
Solution : Let $u = x^2$, so that $du = 2xdx$ or $xdx = \frac{1}{2}du$. Making

these substitutions, we obtain

$$\begin{aligned}\int xsexx^2dx &= \frac{1}{2}\int secudu \\ &= \frac{1}{2}ln|secu + tanu| + C \\ &= \frac{1}{2}ln|secx^2 + tanx^2| + C\end{aligned}$$

Theorem 4. *a.* $\lim_{x\to\infty} lnx = \infty$
b. $\lim_{x\to 0^+} lnx = -\infty$

Proof. a. By Law of logarithms, we have $ln2^n = nln2$ for any positive integer n. Since $ln2 > 0$, we see that $ln2^n \to \infty$ as $n \to \infty$. But lnx is an increasing function, so

$$\lim_{x\to\infty} lnx = \infty$$

b. Let $t = \frac{1}{x}$. Then $t \to \infty$ as $x \to 0^+$. Therefore, using part (a), we have

$$\lim_{x\to 0^+} lnx = \lim_{t\to\infty} ln(\frac{1}{t}) = \lim_{t\to\infty}(-lnt) = \infty$$

□

1.2 Inverse Functions

A function that undoes, or inverts, the effect of a function f is called the inverse of f. Many common functions, though not all, are paired with an inverse. Important inverse functions often show up in applications. Inverse functions also play a key role in the development and properties of the exponential functions. To have an inverse, a function must possess a special property over its domain.

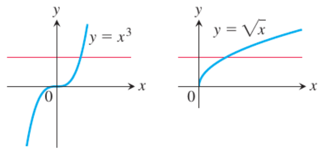
Definition 3. *A function $f(x)$ is one-to-one on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.*

Examples :

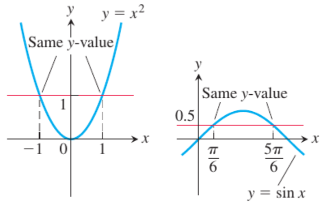
1. $f(x) = \sqrt{x}$ is one-to-one on any domain of non negative numbers because $\sqrt{x_1} \neq \sqrt{x_2}$ whenever $x_1 \neq x_2$.

2. $g(x) = sin(x)$ is not one-to-one on the interval $[0, \pi]$ because $sin(\pi/6) = sin(5\pi/6)$. In fact, for each element x_1 in the sub interval $[0, \pi/2)$ there is a corresponding element x_2 in the sub interval $(\pi/2, \pi]$ satisfying $sin(x_1) = sin(x_2)$. The sine function is one-to-one on $[0, \pi/2]$, however, because it is an increasing function on $[0, \pi/2]$ and therefore gives distinct outputs for distinct inputs in that interval.

The Horizontal Line Test for One-to-One Functions : *A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.*



(a) One-to-one: Graph meets each horizontal line at most once.



(b) Not one-to-one: Graph meets one or more horizontal lines more than once.

Definition 4. *Suppose that f is a one-to-one function on a domain D with range R. The inverse function f^{-1} is defined by $f^{-1}(b) = a$ if $f(a) = b$. The domain of f^{-1} is R and the range of f^{-1} is D.*

Example : Suppose a one-to-one function $y = f(x)$ is given by a table of values

x	1	2	3	4	5	6	7	8
f(x)	3	4.5	7	10.5	15	20.5	27	34.5

Solution : Then the table of values of $x = f^{-1}(y)$ is as given below

y	3	4.5	7	10.5	15	20.5	27	34.5
$f^{-1}(y)$	1	2	3	4	5	6	7	8

Composing a function and its inverse has the same effect as doing nothing i.e
$(f^{-1} \circ f)(x) = x$, for all x in the domain of f
$(f \circ f^{-1})(y) = y$, for all y in the domain of f^{-1} (or range of f).
Keep in mind that only a one-to-one function can have an inverse. The reason is that if $f(x_1) = y$ and $f(x_2) = y$ for two distinct inputs x_1 and x_2 , then there is no way to assign a value to $f^{-1}(y)$ that satisfies both $f^{-1}(f(x_1)) = x_1$ and $f^{-1}(f(x_2)) = x_2$. A function that is increasing on an interval satisfies the inequality $f(x_2) > f(x_1)$ when $x_2 > x_1$, so it is one-to-one and has an inverse. A function that is decreasing on an interval also has an inverse. Functions that are neither increasing nor decreasing may still be one- to-one and have an inverse. An example is the function $f(x) = 1/x$ for $x \neq 0$ and $f(0) = 0$, defined on $(-\infty, \infty)$ and passing the horizontal line test.

Finding the Inverse :

We want to set up the graph of f^{-1} so that its input values lie along the x-axis, as is usually done for functions, rather than on the y-axis. To achieve this we interchange the x- and y-axes by reflecting across the 45 degree line $y = x$. After this reflection we have a new graph that represents f^{-1}. The value of $f^{-1}(x)$ can now be read from the graph in the usual way, by starting with a point x on the x-axis, going vertically to the graph, and then horizontally to the y-axis to get the value of $f^{-1}(x)$.
The process of passing from f to f^{-1} can be summarized as a two-step procedure.

1. Solve the equation $y = f(x)$ for x. This gives a formula $x = f^{-1}(y)$ where x is expressed as a function of y.
2. Interchange x and y, obtaining a formula $y = f^{-1}(x)$ where f^{-1} is expressed in the conventional format with x as the independent variable and y as the dependent variable.

Example : Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of x.
Solution : 1. Solve for x in terms of y:
$y = \frac{1}{2}x + 1$
$2y = x + 2$
$x = 2y - 2.$
2. Interchange x and y: $y = 2x - 2$.
The inverse function of f is $f^{-1}(x) = 2x - 2$.

Theorem 5. ***(The Derivative Rule for Inverses)*** *If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain (the range of f). The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:*

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

or

$$\frac{df^{-1}}{dx}\Big|_{x=b} = \frac{1}{\frac{df}{dx}\Big|_{x=f^{-1}(b)}}$$

Theorem 1 makes two assertions. The first of these has to do with the conditions under which f^{-1} is differentiable; the second assertion is a formula for the derivative of f^{-1} when it exists. The second assertion can be easily proved (hint : start with $f(f^{-1}(x)) = x$ and take the derivative wrt x).

Example : Let $f(x) = x^3 - 1$, for $x > 0$.(x). Find the value of df^{-1}/dx at $x = 6 = f(2)$ without finding the formula for f^{-1}.

Solution : We apply Theorem 1 to obtain the value of the derivative of f^{-1} at $x = 6$:

$$\frac{df}{dx}|_{x=2} = 3x^2|_{x=2} = 12$$

$$\frac{df^{-1}}{dx}|_{x=f(2)} = \frac{1}{\frac{df}{dx}|_{x=2}} = \frac{1}{12}$$

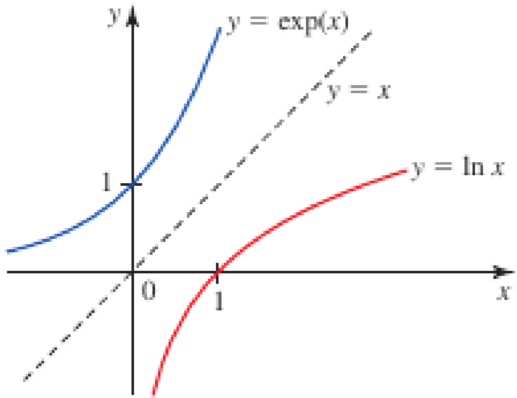
1.3 Exponential functions :

We saw that the natural logarithm function defined by $y = lnx$ is continuous and increasing on the interval $(0, \infty)$. Also, lnx is one-to-one on $(0, \infty)$ and, hence, has an inverse. This inverse function is called the natural exponential function and is defined as follows.

Definition 5. *The natural exponential function, denoted by exp, is the function satisfying the conditions:*
1. $ln(expx) = x$ for all $x \in (\infty, \infty)$
2. $exp(lnx) = x$ for all $x \in (0, \infty)$
Equivalently, $exp(x) = y$ if and only if $lny = x$.

That the domain of exp is $(-\infty, \infty)$ and its range is $(0, \infty)$ follows because the range of ln is $(-\infty, \infty)$ and its domain is

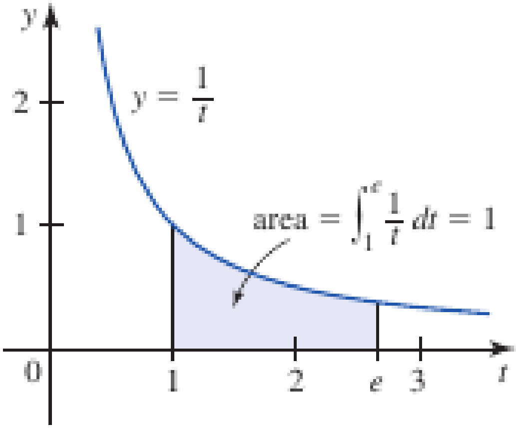
$(0, \infty)$. The graph of $y = exp(x)$ can be obtained by reflecting the graph of $y = lnx$ about the line $y = x$.



Recall that the natural logarithmic function ln is continuous and one-to-one and that its range is (∞, ∞). Therefore, by the Intermediate Value Theorem there must be a unique real number x_0 such that $lnx_0 = 1$. Let's denote x_0 by e. We will formally define e as follows.

Definition 6. *(**The number** e)The number e is the number such that*

$$lne = \int_t^e \frac{1}{t} dt = 1$$



The graph above gives a geometric representation of the number e. It should be noted that e is an irrational number and has the approximate value of 2.718281828.

Natural exponential function : For a real number x, we have

$$lne^x = xlne = x(1) = x$$

Given this, let define natural exponential function.

Definition 7. *The natural exponential function, exp, is defined by the rule*

$$exp(x) = e^x$$

In view of this, we have the following theorem, which gives us another way of expressing the fact that exp and ln are inverse functions.

Theorem 6. *a.* $lne^x = x$*, for* $x \in (-\infty, \infty)$
b. $e^{lnx} = x$*, for* $x \in (0.\infty)$

Properties of natural exponential function :

1. The domain of $f(x) = e^x$ is $(-\infty, \infty)$, and its range is $(0, \infty)$.
2. The function $f(x) = e^x$ is continuous and increasing on $(-\infty, \infty)$.
3. The graph of $f(x) = e^x$ is concave upward on $(-\infty, \infty)$.
4. $\lim_{x\to-\infty} e^x = 0$ and $\lim_{x\to\infty} e^x = \infty$.

Theorem 7. ***(Laws of exponents)*** *Let* x *and* y *be real numbers and* r *be a rational number. Then*
a. $e^x e^y = e^{x+y}$
b. $\frac{e^x}{e^y} = e^{x-y}$
c. $(e^x)^r = e^{rx}$

Proof. We will prove Law (a). The proofs of the other two laws are similar and is left yo the reader. We have

$$ln(e^x e^y) = lne^x + lne^y = x + y = lne^{x+y}$$

Since the natural logarithmic function is one-to-one, we see that

$$e^x e^y = e^{x+y}$$

□

Theorem 8. ***(The derivatives of exponential functions)*** *Let* u *be a differentiable function of* x*. Then*
a. $\frac{d}{dx}e^x = e^x$
b. $\frac{d}{dx}e^u = e^u\frac{du}{dx}$

Proof. a. Let $y = e^x$, so that $lny = x$. Differentiating both sides of the last equation implicitly with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = 1 \quad or \quad \frac{dy}{dx} = y = e^x$$

b. This follows from part (a) by using the Chain Rule. □

Example : Find the derivative of

a. $f(x) = e^{-x^2}$

b. $y = ln(e^{2x} + e^{-2x})$.

Solution : a. $f'(x) = \frac{d}{dx}e^{-x^2} = e^{-x^2}\frac{d}{dx}(-x^2) = -2xe^{-x^2}$

b.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}ln(e^{2x} + e^{-2x}) \\ &= \frac{1}{e^{2x} + e^{-2x}}\frac{d}{dx}(e^{2x} + e^{-2x}) \\ &= \frac{1}{e^{2x} + e^{-2x}}(2e^{2x} - 2e^{-2x}) \\ &= \frac{2(e^{2x} - e^{-2x})}{e^{2x} + e^{-2x}}\end{aligned}$$

Theorem 9. ***(Integration of natural exponential function)*** *Since the derivative of the natural exponential function is the function itself, the following theorem is immediate. Let u be a differentiable function of x. Then*

$$\int e^u du = e^u + C$$

Example : Find

a. $\int e^{5x}dx$

b. $\int_0^1 \frac{e^x}{1+e^x}dx$

Solution : a. Let $u = 5x$, so that $du = 5dx$, or $dx = \frac{1}{5}du$. Making these substitutions, we obtain

$$\int e^{5x}dx = \frac{1}{5}\int e^u du = \frac{1}{5}e^u + C = \frac{1}{5}e^{5x} + C$$

b. Let $u = 1 + e^x$, so that $du = e^x dx$. If $x = 0$, then $u = 2$; and if $x = 1$, then $u = 1 + e$. This gives the lower and upper limits of integration with respect to u. We have

$$\int_0^1 \frac{e^x}{1+e^x}dx = \int_2^{1+e} \frac{1}{u}du = [lnu]_2^{1+e} = ln(1+e) - ln2 \approx 0.620$$

1.4 General exponential and logarithmic functions

Exponential functions with base a : The natural exponential function defined by $f(x) = e^x$ has base e. We will now consider exponential functions that have bases other than e.

Definition 8. *Let a be a positive real number with $a \neq 1$. The exponential function with base a is the function f defined by*

$$f(x) = a^x = e^{xlna}$$

Theorem 10. *Let a and b be positive numbers. If x and y are real numbers, then*
a. $a^x a^y = a^{x+y}$
b. $(a^x)^y = a^{xy}$
c. $(ab)^x = a^x b^x$

d. $\frac{a^x}{a^y} = a^{x-y}$
e. $(\frac{a}{b})^x = \frac{a^x}{b^x}$

Proof. We will prove the first law and leave the proofs of the other laws as exercises.

$$\begin{aligned} a^x a^y &= e^{xlna} e^{ylna} \\ &= e^{xlna+ylna} \\ &= e^{x+y} lna \\ &= a^{x+y} \end{aligned}$$

□

Theorem 11. ***(Derivatives of*** a^x ***and*** a^u***)*** *Let a be a positive number with $a \neq 1$, and let u be a differentiable function of x. Then*
a. $\frac{d}{dx} a^x = (lna) a^x$
b. $\frac{d}{dx} a^u = (lna) a^u \frac{du}{dx}$

Proof. a. $\frac{d}{dx} a^x = \frac{d}{dx} e^{xlna} = e^{xlna} \frac{d}{dx}(xlna) = e^{xlna}(lna) = (lna) a^x$
b. Follows from the chain rule. □

Example : Find the derivative of
a. $f(x) = 2^x$
b. $y = 10^{cos2x}$
Solution : a. $f'(x) = \frac{d}{dx} 2^x = (ln2) 2^x$
b.

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} 10^{cos2x} \\ &= (ln10) 10^{cos2x} \frac{d}{dx} cos2x \\ &= (ln10) 10^{cos2x} (-sin2x)(2) \\ &= -2(ln10)(sin2x) 10^{cos2x} \end{aligned}$$

Graphs of $\mathbf{y = a^x}$

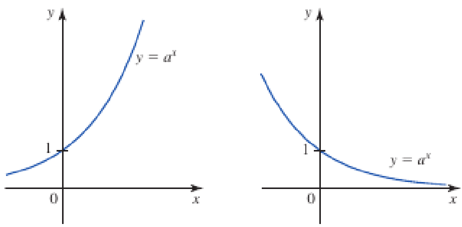
If $a > 1$, then $lna > 0$, and therefore,

$$\frac{d}{dx}(a^x) = a^x lna > 0$$

This shows that the graph of $y = a^x$ is rising on $(-\infty, \infty)$. If $0 < a < 1$, then $lna < 0$, and

$$\frac{d}{dx}(a^x) = a^x lna < 0$$

This implies that if $0 < a < 1$, the graph of $y = a^x$ is falling on $(= \infty, \infty)$. The general shape of the graphs of $y = a^x$ are shown below.



(a) $a > 1$ (b) $0 < a < 1$

Example : Find the derivative of $f(x) = x^x$.

Solution : Let $y = x^x$. Taking the natural logarithm on both sides, we obtain

$$lny = lnx^x = xlnx$$

Differentiating both sides of this equation with respect to x, we obtain

$$\frac{y'}{y} = \frac{d}{dx}(xlnx) = x\frac{d}{dx}(lnx) + (lnx)\frac{d}{dx}(x)$$

Therefore, upon multiplying both sides by y, we obtain

$$y' = (1 + lnx)y = (1 + lnx)x^x$$

Integrating a^x : The formula for integrating an exponential function with base a follows from reversing the differentiation formula. Thus, we have

$$\int a^x dx = \frac{a^x}{lna} + C \quad a > 0 \;\; and \;\; a \neq 1$$

Logarithmic functions with base a : If a is a positive real number with $a \neq 1$, then the function f defined by $f(x) = a^x$ is one-to-one on $(-\infty, \infty)$, and its range is $(0, \infty)$. Therefore, it has an inverse on $(0, \infty)$. This function is called the logarithmic function with base a and is denoted by log_a.

Definition 9. *The logarithmic function with base* a*, denoted by* log_a*, is the function satisfying the relationship*

$$y = log_a x \quad if \; and \; only \; if \quad x = a^y$$

Change of base formula :

$$log_a x = \frac{lnx}{lna} \;\; a > 0 \;\; and \;\; a \neq 1$$

Theorem 12. ***(The power rule)*** *If n is a real number, then*

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Proof. Let $y = x^n$ and consider the equation

$$|y| = |x^n| = |x|^n \quad x \neq 0$$

Taking the natural logarithm on both sides of the equation, we obtain

$$ln|y| = nln|x|$$

which, upon differentiation with respect to x, yields

$$\frac{y'}{y} = \frac{n}{x}$$

or

$$y' = \frac{ny}{x} = \frac{nx^n}{x} = nx^{n-1}$$

$\square$

Theorem 13. ***(Derivatives of the logarithmic function with base***
a) Let u be a differentiable function of x. Then
a. $\frac{d}{dx}log_a|x| = \frac{1}{xlna} \quad x \neq 0$
b. $\frac{d}{dx}log_a|u| = \frac{1}{ulna}.\frac{du}{dx} \quad u \neq 0$

Example : Find the derivative of $f(x) = x^2log(e^{2x} + 1)$.

Solution : Using the product rule, we obtain

$$\begin{aligned} f'(x) &= \frac{d}{dx}[x^2 log(e^{2x}+1)] \\ &= [\frac{d}{dx}(x^2)]log(e^{2x}+1) + x^2\frac{d}{dx}log(e^{2x}+1) \\ &= 2xloh(e^{2x}+1) + \frac{x^2}{(e^{2x}+1)ln10}.\frac{d}{dx}(e^{2x}+1) \\ &= 2xlon(e^{2x}+1) + \frac{2x^{2x}}{(e^{2x}+1)ln10} \end{aligned}$$

The definition of the number e as a limit : If we use the definition of the derivative as a limit to compute $f'(1)$, where $f(x) = lnx$, we obtain

$$\begin{aligned} f'(1) &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} \\ &= \lim_{h\to 0}\frac{ln(1+h)-ln(1)}{h} = \lim_{h\to 0}\frac{ln(1+h)}{h} \\ &= \lim_{h\to 0} ln(1+h)^{1/h} \\ &= ln[\lim_{h\to 0}(1+h)^{1/h}] \end{aligned}$$

But

$$f'(1) = [\frac{d}{dx}lnx]_{x=1} = [\frac{1}{x}]_{x=1} = 1$$

Thus,

$$ln[\lim_{h\to 0}(1+h)^{1/h}] = 1$$

or

$$\lim_{h\to 0}(1+h)^{1/h} = e$$

Above equation is sometimes used to define the number e. Another equivalent definition of e is:

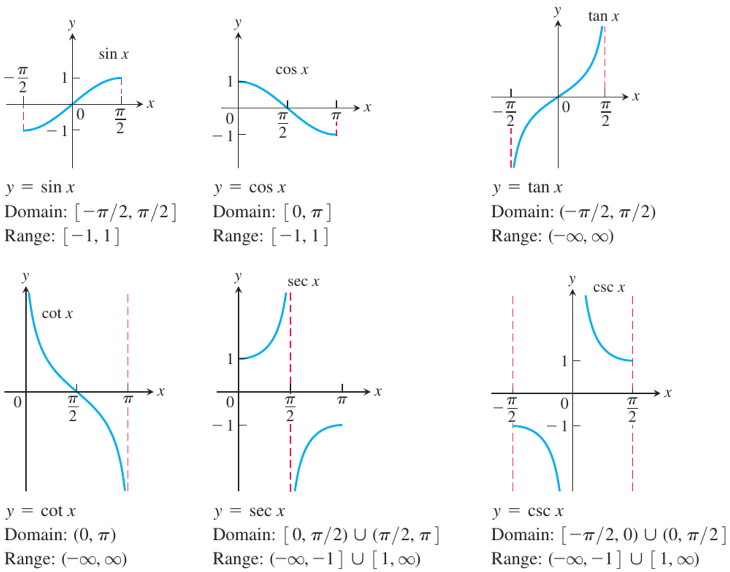
$$\lim_{n\to\infty}(1+\frac{1}{n})^n = e$$

1.5 Inverse trigonometric functions :

Inverse trigonometric functions arise when we want to calculate angles from side measurements in triangles. They also provide useful anti derivatives and appear frequently in the solutions of differential equations.

The six basic trigonometric functions are not one-to-one since their values repeat periodically. However, we can restrict their domains to intervals on which they are one-to-one. The sine function increases from -1 at $x = -\pi/2$ to $+1$ at $x = \pi/2$. By restricting domain to $[-\pi/2, \pi/2]$ we make it one-one, so it has an inverse which trigonometric is called $arcsinx$. Similar domain restrictions can be applied to all six trigonometric functions.

Domain restrictions that make the trigonometric functions one-to-one



$y = \sin x$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$

$y = \cos x$
Domain: $[0, \pi]$
Range: $[-1, 1]$

$y = \tan x$
Domain: $(-\pi/2, \pi/2)$
Range: $(-\infty, \infty)$

$y = \cot x$
Domain: $(0, \pi)$
Range: $(-\infty, \infty)$

$y = \sec x$
Domain: $[0, \pi/2) \cup (\pi/2, \pi]$
Range: $(-\infty, -1] \cup [1, \infty)$

$y = \csc x$
Domain: $[-\pi/2, 0) \cup (0, \pi/2]$
Range: $(-\infty, -1] \cup [1, \infty)$

Since these restricted functions are not one-one, we cam defined their inverses and denoted as follows:

$$y = sin^{-1}(x) \;\; or \;\; y = arcsin(x)$$

$$y = cos^{-1}(x) \;\; or \;\; y = arccos(x)$$

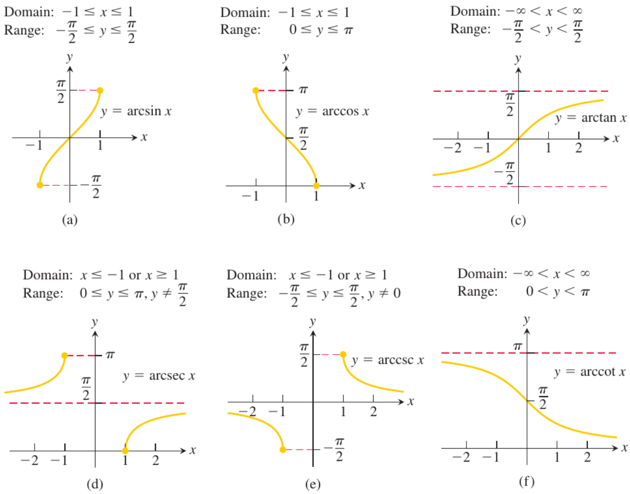
$$y = tan^{-1}(x) \;\; or \;\; y = arctan(x)$$

$$y = csc^{-1}(x) \;\; or \;\; y = arcsc(x)$$

$$y = sec^{-1}(x) \;\; or \;\; y = arcsec(x)$$

$$y = cot^{-1}(x) \;\; or \;\; y = arccot(x)$$

Graphs of basic inverse trigonometric functions:



Definition 10. *$y = arcsinx$ is the number in $[-\pi/2, \pi/2]$ for which $siny = x$.*
$y = arccosx$ is the number in $[0, \pi]$ for which $cosy = x$.

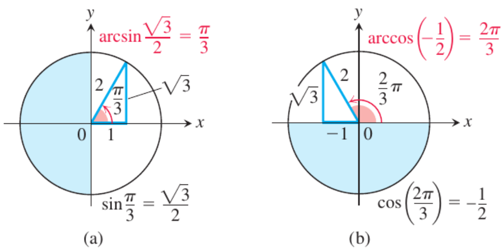
The graph of $y = arcsinx$ as shown in the figure above, is symmetric about the origin (it lies along the graph of $x = siny$). The $arcsine$ is therefore an odd function i.e. $arcsin(-x) = -arcsinx$. The graph of $y = arccosx$ has no such symmetry.

Example : Evaluate
a. $arcsin(\sqrt{3}/2)$
b. $arccos(-1/2)$
Solution : a. $sin(\pi/3) = \sqrt{3}/2$ and $\pi/3$ belongs to the range $[-\pi/2, \pi/2]$ of the $arcsine$ function. Therefore, $arcsin(\sqrt{3}/2) = \pi/3$.
b. $cos(2\pi/3) = -1/2$ and $2\pi/3$ belongs to the range $[0, \pi]$ of the

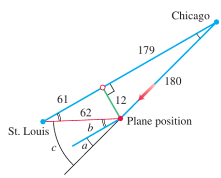
arccosine function. Therefore, $arccos(-1/2) = 2\pi/3$.

Using the procedure as above, we can find common values for *arcsin* and *arccos* functions.

x	$\arcsin x$	$\arccos x$
$\sqrt{3}/2$	$\pi/3$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$	$\pi/4$
$1/2$	$\pi/6$	$\pi/3$
$-1/2$	$-\pi/6$	$2\pi/3$
$-\sqrt{2}/2$	$-\pi/4$	$3\pi/4$
$-\sqrt{3}/2$	$-\pi/3$	$5\pi/6$



Example : During a 240 mi airplane flight from Chicago to St. Louis, after flying 180 mi the navigator determines that the plane is 12 mi off course, as shown in Figure 7.26. Find the angle a for a course parallel to the original correct course, the angle b, and the drift correction angle c = a + b.



Solution : Using the Pythagorean theorem, we compute an approximate hypothetical flight distance of 179 mi, had the plane been flying along the original correct course. Knowing the flight distance from Chicago to St. Louis, we next calculate the remaining leg of the original course to be 61 mi. Applying the Pythagorean theorem again then gives an approximate distance of 62 mi from the position of the plane to St. Louis. Finally, we see that $180 sina = 12$ and $62 sinb = 12$, so

$$a = arcsin(12/180) \approx 0.067 radian \approx 3.8^\circ$$

$$b = arcsin(12/62) \approx 0.195 radian \approx 11.2^\circ$$

Thus, $c = a + b \approx 15^\circ$.

Identities Involving Inverse Trigonometric functions :

1. $arccosx + arccos(-x) = \pi$
or
$arccos(-x) = \pi - arccosx.$
2. $arccotx = \pi/2 - arctanx.$
3. $arccscx = \pi/2 - arcsecx.$
4. $arcsinx + arccosx = \pi/2.$

Definition 11. *1.* $y = arctanx$ *is the number in* $(-\pi/2, \pi/2)$ *for which* $tany = x$.
2. $y = arccotx$ *is the number in* $(0, \pi)$ *for which* $coty = x$.
3. $y = arcsecx$ *is the number in* $(0, \pi/2) \cup (\pi/2, \pi)$ *for which*

$secy = x$.
4. $y = arccscx$ *is the number in* $(-\pi/2, 0) \cup (0, \pi/2)$ *for which* $cscy = x$.

Derivative of Inverse trigonomnetric functions

Derivative of $y = arcsin(x)$.

$$y = arcsin(x)$$

$$x = sin(y)$$

Derivating wrt x,

$$1 = cos(y)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{cos(y)}$$

We have $x = sin(y)$, that implies $cos(y) = \sqrt{1-x^2}$. Substituting into the above equation, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Derivative of $y = arctan(x)$.

$$y = arctan(x)$$

$$x = tan(y)$$

Derivating wrt x,

$$1 = sec^2(y)\frac{dy}{dx}$$

$x = tan(y)$, that implies $sec^2(y) = 1 + x^2$. Substituting into the above equation, we get

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

We can find the derivatives of remaining inverse trigonometric functions in similar fashion. The table below summarises the derivatives,

1. $\dfrac{d(\arcsin u)}{dx} = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}, \quad |u| < 1$
2. $\dfrac{d(\arccos u)}{dx} = -\dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}, \quad |u| < 1$
3. $\dfrac{d(\arctan u)}{dx} = \dfrac{1}{1+u^2}\dfrac{du}{dx}$
4. $\dfrac{d(\operatorname{arccot} u)}{dx} = -\dfrac{1}{1+u^2}\dfrac{du}{dx}$
5. $\dfrac{d(\operatorname{arcsec} u)}{dx} = \dfrac{1}{|u|\sqrt{u^2-1}}\dfrac{du}{dx}, \quad |u| > 1$
6. $\dfrac{d(\operatorname{arccsc} u)}{dx} = -\dfrac{1}{|u|\sqrt{u^2-1}}\dfrac{du}{dx}, \quad |u| > 1$

Integration formulas

The derivative formulas above yield three useful integration formulas in the table below. The formulas are readily verified by differentiating the functions on the right-hand sides.

The following formulas hold for any constant $a > 0$.

1. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$ (Valid for $u^2 < a^2$)
2. $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$ (Valid for all u)
3. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$ (Valid for $|u| > a > 0$)

Example :

a.

$$\int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} \frac{dx}{\sqrt{1-x^2}} = sin^{-1}(x)|_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}}$$

$$= sin^{-1}(\frac{\sqrt{3}}{2}) - sin^{-1}(\frac{\sqrt{2}}{2})$$

$$= \frac{\pi}{3} - \frac{\pi}{4}$$

$$= \frac{\pi}{12}$$

b.

$$\int \frac{dx}{\sqrt{3-4x^2}} = \frac{1}{2}\int \frac{du}{\sqrt{a^2-u^2}}. \quad set\ a = \sqrt{3}, u = 2x$$

$$= \frac{1}{2}sin^{-1}(\frac{u}{a}) + C$$

$$= \frac{1}{2}sin^{-1}(\frac{2x}{\sqrt{3}}) + C$$

1.6 Hyperbolic functions

The analysis of many problems in engineering and mathematics involves combinations of exponential functions of the form e^{cx} and e^{cx}, where c is a constant. Because combinations of these functions arise so frequently in mathematics and its applications, they have been given special names. These combinations—the hyperbolic sine, the hyperbolic cosine, the hyperbolic tangent, and so on—are referred to as hyperbolic functions and are so called because they have many properties in common with the trigonometric functions.

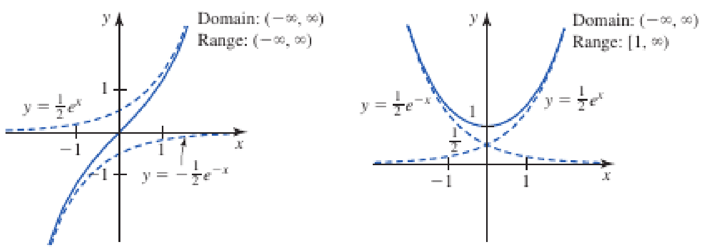
Definition 12. ***(Hyperbolic functions)***

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} \qquad \operatorname{csch}(x) = \frac{1}{\sinh(x)}, \quad x \neq 0$$

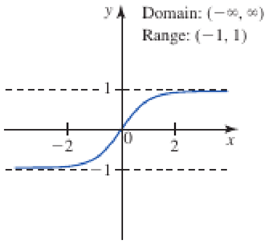
$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} \qquad \coth(x) = \frac{\cosh(x)}{\sinh(x)}, \quad x \neq 0$$

The graphs of the Hyperbolic functions : The graph of $y = \sinh(x)$ can be drawn by first sketching the graphs of $y = 1/2e^x$ and $y = -1/2e^{-}x$ and then adding the y-coordinates of the points on these graphs corresponding to each x to obtain the y-coordinates of the points on $y = \sinh(x)$. Similarly, the graph of $y = \cosh(x)$ can be drawn by first sketching the graphs of $y = 1/2e^x$ and $y = 1/2e^{-x}$ and then adding the y-coordinates of the points on these graphs corresponding to each x to obtain the y-coordinates of the points on $y = \cosh(x)$.

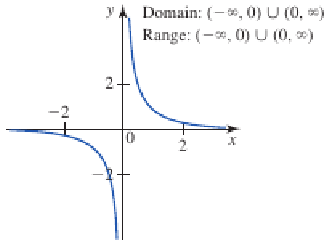


(a) $y = \sinh x = \frac{e^x - e^{-x}}{2}$ (b) $y = \cosh x = \frac{e^x + e^{-x}}{2}$

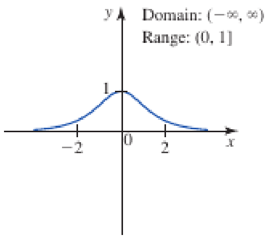
The graphs of the other four hyperbolic functions are shown below.



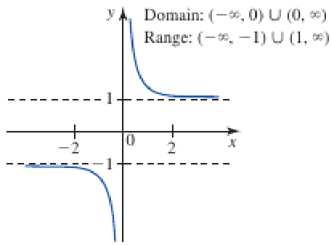
(a) $y = \tanh x = \dfrac{\sinh x}{\cosh x}$



(b) $y = \operatorname{csch} x = \dfrac{1}{\sinh x}$



(c) $y = \operatorname{sech} x = \dfrac{1}{\cosh x}$



(d) $y = \coth x = \dfrac{1}{\tanh x}$

Hyperbolic identities : The hyperbolic functions satisfy certain identities that look very much like those satisfied by trigonometric functions. The list of frequently used hyperbolic identities is given below.

Theorem 14. *Hyperbolic identities*

a. $\sinh(-x) = -\sinh(x)$

b. $\cosh(-x) = \cosh(x)$

c. $\cosh^2(x) - \sinh^2(x) = 1$

d. $\operatorname{sech}^2(x) = 1 - \tanh^2(x)$

e. $\sinh(x + y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$

f. $\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$

g. $\sinh(2x) = 2\sinh(x)\cosh(x)$
h. $\cosh(2x) = \cosh^2(x) + \sinh^2(x)$
i. $\cosh^2(x) = \frac{1}{2}(1 + \cosh(2x))$
j. $\sinh^2(x) = \frac{1}{2}(-1 + \cosh(2x))$

Proof. We will discuss the proof of (a) and (c). rest is left to the reader as exercise (Hint: use the definition of hyperbolic functions!)
a. $\sinh(-x) = \frac{e^{-x}-e^{-(-x)}}{2} = \frac{e^{-x}-x^{x}}{2} = -\frac{e^{x}-e^{-x}}{2} = -\sinh(x)$
c.

$$\begin{aligned}\cosh^2(x) - \sinh^2(x) &= (\frac{e^x + e^{-x}}{2})^2 - (\frac{e^x - e^{-x}}{2})^2 \\ &= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} \\ &= 1\end{aligned}$$

□

Derivatives and integrals of hyperbolic functions : Since the hyperbolic functions are defined in terms of e^x and e^{-x}, their derivatives are easily computed. For example,

$$\frac{d}{dx}(\sinh(x)) = \frac{d}{dx}(\frac{e^x - e^{-x}}{2}) = \frac{e^x + e^{-x}}{2} = \cosh(x)$$

. The table below summarises the differentiation formulas together with the corresponding integration formulas for the six hyperbolic functions.

Derivatives and Integrals of Hyperbolic Functions

$\dfrac{d}{dx}(\sinh u) = (\cosh u)\dfrac{du}{dx}$	$\int \cosh u\, du = \sinh u + C$
$\dfrac{d}{dx}(\cosh u) = (\sinh u)\dfrac{du}{dx}$	$\int \sinh u\, du = \cosh u + C$
$\dfrac{d}{dx}(\tanh u) = (\operatorname{sech}^2 u)\dfrac{du}{dx}$	$\int \operatorname{sech}^2 u\, du = \tanh u + C$
$\dfrac{d}{dx}(\operatorname{csch} u) = -(\operatorname{csch} u \coth u)\dfrac{du}{dx}$	$\int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$
$\dfrac{d}{dx}(\operatorname{sech} u) = -(\operatorname{sech} u \tanh u)\dfrac{du}{dx}$	$\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$
$\dfrac{d}{dx}(\coth u) = -(\operatorname{csch}^2 u)\dfrac{du}{dx}$	$\int \operatorname{csch}^2 u\, du = -\coth u + C$

Example : Find the derivative of $\cosh^2(ln2x)$.

Solution :

$$\begin{aligned}\frac{d}{dx}\cosh^2(ln2x) &= 2\cosh(ln2x)\frac{d}{dx}\cosh(ln2x)\\ &= 2\cosh(ln2x)\sinh(ln2x)\frac{d}{dx}ln2x\\ &= \frac{2}{x}\cosh(ln2x)\sinh(ln2x)\end{aligned}$$

Example : Find $\int \cosh^2(3x)\sinh(3x)dx$.

Solution : Let $u = 3x$ so that $du = 3dx$ or $dx = \frac{1}{3}du$. Then

$$\int \cosh^2(3x)\sinh(3x)dx = \frac{1}{3}\int \cosh^2(u)\sinh(u)du$$

Next, let $v = \cosh(u)$ so that $dv = \sinh(u)du$. Then

$$\frac{1}{3}\int \cosh^2(u)\sinh(u)du = \frac{1}{3}\int v^2 dv = \frac{1}{9}v^3 + C$$

So,

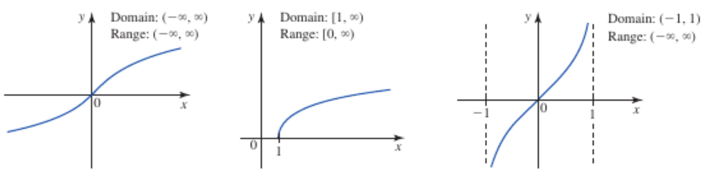
$$\int \cosh^2(3x)\sinh(3x)dx = \frac{1}{9}\cosh^3(3x) + C$$

Inverse Hyperbolic functions : Notice that both $\sinh(x)$ and $\tanh(x)$ are one-to-one on $(-\infty, \infty)$ and hence have inverse functions that we denote by $\sinh^{-1}(x)$ and $\tanh^{-1}(x)$ respectively. Also, $\cosh(x)$ is one-to-one on $[0, \infty)$, so, if restricted to this domain, it has an inverse, $\cosh^{-1}(x)$. By examining the graphs of the other hyperbolic functions and making the necessary restrictions on their domains, we are able to define the other inverse hyperbolic functions.

DEFINITIONS **Inverse Hyperbolic Functions**

			Domain
$y = \sinh^{-1} x$	if and only if	$x = \sinh y$	$(-\infty, \infty)$
$y = \cosh^{-1} x$	if and only if	$x = \cosh y$	$[1, \infty)$
$y = \tanh^{-1} x$	if and only if	$x = \tanh y$	$(-1, 1)$
$y = \operatorname{csch}^{-1} x$	if and only if	$x = \operatorname{csch} y$	$(-\infty, 0) \cup (0, \infty)$
$y = \operatorname{sech}^{-1} x$	if and only if	$x = \operatorname{sech} y$	$(0, 1]$
$y = \coth^{-1} x$	if and only if	$x = \coth y$	$(-\infty, -1) \cup (1, \infty)$

The graphs of $y = \sinh^{-1}(x)$, $y \cosh^{-1}(x)$, and $y = \tanh^{-1}(x)$ are shown below.



(a) $y = \sinh^{-1} x$ (b) $y = \cosh^{-1} x$ (c) $y = \tanh^{-1} x$

Example : Show that $\sinh^{-1}(x) = ln(x + \sqrt{(}x^2 + 1))$.
Solution : Let $y = \sinh^{-1}(x)$. Then

$$x = \sinh(y) = \frac{e^y - e^{-y}}{2}$$

or

$$e^y - 2x - e^{-y} = 0$$

On multiplying both sides of this equation by e^y, we obtain

$$e^{2y} - 2xe^y - 1 = 0$$

which is a quadratic in e^y. Using the quadratic formula, we have

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}$$

Only the root $x + \sqrt{x^2 + 1}$ is admissible since $x - \sqrt{x^2 + 1} < 0$ but $e^y > 0$. Therefore, we have

$$e^y = x + \sqrt{x^2 + 1}$$

or

$$y = ln(x + \sqrt{x^2 + 1})$$

that is,

$$\sinh^{-1}(x) = ln(x + \sqrt{x^2 + 1})$$

In this similar manner, we can find out the representations of the other inverse hyperbolic functions.

Representations of Inverse Hyperbolic Functions in Terms of Logarithmic Functions

$$\begin{aligned}
\sinh^{-1}(x) &= ln(x+\sqrt{x^2+1}), \quad x \in (-\infty, \infty) \\
\cosh^{-1}(x) &= ln((x+\sqrt{x^2-1}), \quad x \in [1, \infty) \\
\tanh^{-1}(x) &= \frac{1}{2} ln(\frac{1+x}{1-x}), \quad x \in (-1, 1)
\end{aligned}$$

Derivatives of Inverse Hyperbolic functions : The derivatives of the inverse hyperbolic functions can be found by differentiating the function in question directly. For example,

$$\begin{aligned}
\frac{d}{dx}\sinh^{-1}(x) &= \frac{d}{dx} ln(x+\sqrt{x^2+1}) \\
&= \frac{1}{x+\sqrt{x^2+1}}[1+\frac{1}{2}(x^2+1)^{-1/2}(2x)] \\
&= \frac{1}{x+\sqrt{x^2+1}} . \frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}} \\
&= \frac{1}{\sqrt{x^2+1}}
\end{aligned}$$

Alternatively, $y = \sinh^{-1}(x)$ if and only if $x = \sinh(y)$

Differentiating this last equation implicitly with respect to x, we obtain

$$\frac{d}{dx}(x) = \frac{d}{dx}(\sinh(y))$$

$$1 = (\cosh(y))\frac{dy}{dx}$$

or

$$\frac{dy}{dx} = \frac{1}{\cosh(y)} = \frac{1}{\sqrt{\sinh^2(y)+1}} = \frac{1}{\sqrt{x^2+1}}$$

Using techniques above, we obtain the following formulas for differentiating the inverse hyperbolic functions.

Derivatives of Inverse Hyperbolic Functions

$$\frac{d}{dx}\sinh^{-1}u = \frac{1}{\sqrt{u^2+1}}\frac{du}{dx} \qquad \frac{d}{dx}\cosh^{-1}u = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$$

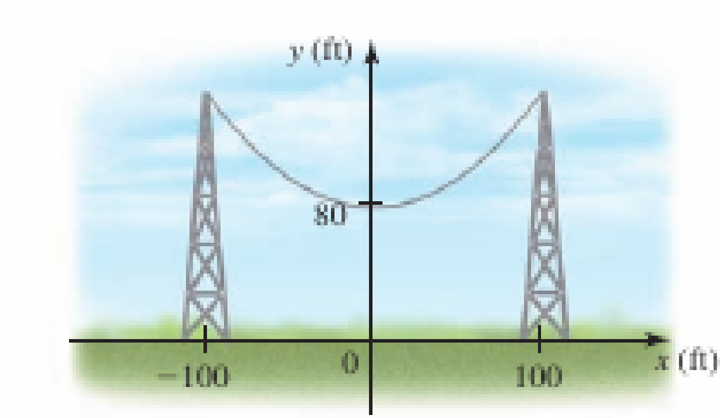
$$\frac{d}{dx}\tanh^{-1}u = \frac{1}{1-u^2}\frac{du}{dx} \qquad \frac{d}{dx}\operatorname{csch}^{-1}u = -\frac{1}{|u|\sqrt{u^2+1}}\frac{du}{dx}$$

$$\frac{d}{dx}\operatorname{sech}^{-1}u = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx} \qquad \frac{d}{dx}\coth^{-1}u = \frac{1}{1-u^2}\frac{du}{dx}$$

Example : A power line is suspended between two towers as depicted in Figure below. The shape of the cable is a catenary with equation

$$y = 80\cosh\left(\frac{x}{80}\right) \qquad -100 \leq x \leq 100$$

where x is measured in feet. Find the length of the cable.



Solution : Taking advantage of the symmetry of the situation, we see that the required length is given by

$$L = 2\int_0^{100} \sqrt{1+(\frac{dy}{dx})^2}dx$$

But

$$\frac{dy}{dx} = \frac{d}{dx}[80\cosh(\frac{x}{80})] = 80\sinh(\frac{x}{80}).\frac{d}{dx}(\frac{x}{80}) = \sinh(\frac{x}{80})$$

So,

$$\begin{aligned}\sqrt{1+(\frac{dy}{dx})^2} &= \sqrt{1+\sinh^2(\frac{x}{80})} = \sqrt{1+\cosh^2(\frac{x}{80})-1} \\ &= \sqrt{\cosh^2(\frac{x}{80})} = \cosh(\frac{x}{80})\end{aligned}$$

Therefore,

$$\begin{aligned}L &= 2\int_0^{100} \cosh(\frac{x}{80})dx \\ &= 2[80\sinh(\frac{x}{80})]_0^{100} \\ &= 160\sinh(\frac{100}{80}) = 160\sinh(\frac{5}{4}) \\ &\approx 256ft.\end{aligned}$$

1.7 Indeterminate forms and l'Hopital's rule

If the $\lim_{x\to a} f(x) = 0$ and and $\lim_{x\to a} g(x) = 0$, then the limit

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

is called an indeterminate form of type $0/0$. The undefined expression $0/0$ does not provide us with a definitive answer concerning the existence of the limit or its value, if the limit exists.So, given an indeterminate form of the type $0/0$, we want to see if there is a more general and efficient method for resolving whether the limit

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

exists, and if so, what is the limit?

The indeterminate forms $0/0$ and ∞/∞

Theorem 15. ***(l'Hopital's** Rule) Suppose that f and g are differentiable on an open interval I that contains a, with the possible exception of a itself, and $g'(x) \neq 0$ for all x in I. If $\lim_{x\to a} \frac{f(x)}{g(x)}$ indeterminate form of the type $0/0$ or ∞/∞ , then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided that the limit on the right-hand side exists or is infinite.

Notes :

1. l'Hopital's Rule is also valid for one-sided limits as well as

limits at infinity or negative infinity; that is, we can replace $x \to a$ by any of the symbols $x \to a^+$, $x \to a^-$, $x \to \infty$, or $x \to -\infty$.

2. Before applying l'Hopital's Rule, check to see that the limit does have one of the indeterminate forms. For example, $cos(x) \to 1$ as $x \to 0^+$, so

$$\lim_{x\to 0^+} \frac{cos(x)}{x} = \infty$$

If we had applied l'Hopital's Rule to evaluate the limit without first ascertaining that it had an indeterminate form, we would have obtained the erroneous result

$$\lim_{x\to 0^+} \frac{cos(x)}{x} = \lim_{x\to 0^+} \frac{-sin(x)}{1} = 0$$

Example : Evaluate $\lim_{x\to 1^+} \frac{sin(\pi x)}{\sqrt{x-1}}$

Solution : We have an indeterminate form of the type 0¿0. Applying l'Hopital's Rule, we obtain

$$\begin{aligned} \lim_{x\to 1^+} \frac{sin(\pi x)}{\sqrt{x-1}} &= \lim_{x\to 1^+} \frac{\pi cos(\pi x)}{\frac{1}{2}(x-1)^{-1/2}} \\ &= \lim_{x\to 1^+} 2\pi(cos(\pi x))\sqrt{x-1} \\ &= 0 \end{aligned}$$

The Indeterminate forms $\infty - \infty$ and $0.\infty$

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit

$$\lim_{x\to a}[f(x0 - g(x)]$$

is said to be an indeterminate form of the type $\infty - \infty$. An indeterminate form of this type can often be expressed as one of the type $0/0$ or ∞/∞ by algebraic manipulation.

Example : Evaluate $\lim_{x\to 0^+}(\frac{1}{x} - \frac{1}{e^x-1})$.
Solution : We have an indeterminate form of the type $\infty - \infty$. By writing the expression as a single fraction, we obtain the indeterminate form of the type $0/0$. This enables us to evaluate the resulting expression using l'Hopital's Rule

$$\begin{aligned}\lim_{x\to 0^+}(\frac{1}{x} - \frac{1}{e^x-1} &= \lim_{x\to 0^+}\frac{e^x - x - 1}{x(e^x-1)}\\ &= \lim_{x\to 0^+}\frac{e^x-1}{e^x - 1 + xe^x)}\\ &= \lim_{x\to 0^+}\frac{e^x}{(x+2)e^x} = \frac{1}{2}\end{aligned}$$

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) \pm\infty$, then $\lim_{x\to a} f(x)g(x)$ is said to be an indeterminate form of the type $0.\infty$. An indeterminate form of this type also can be expressed as one of the type $0/0$ or ∞/∞ by algebraic manipulation.

Example : Evaluate $\lim_{x\to)^+} lnx$.
Solution : We have an indeterminate form of the type $0.\infty$. By writing

$$xlnx = \frac{lna}{\frac{1}{x}}$$

the given limit can be cast in an indeterminate form of the type

∞/∞. Then, applying l'Hopital's Rule, we obtain

$$\lim_{x\to 0^+} xlnx = \lim_{x\to 0^+} \frac{lna}{\frac{1}{x}}$$
$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} (-x) = 0$$

The Indeterminate forms 0^0, ∞^0, and 1^∞

The limit

$$\lim_{x\to a} [f(x)]^{g(x)}$$

is said to be an indeterminate form of the type
1. 0^0 if $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$.
2. ∞^0 if $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$.
3. 1^∞ if $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$.

Example : Evaluate $\lim_{x\to 0^+} x^x$.
Solution : We have an indeterminate form of the type 0^0. Let

$$y = x^x$$

Then

$$lny = lnx^x = xlnx$$

Finally, using the identity $y = e^{lny}$ and the continuity of the exponential function, we have

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{lny} = e^{\lim_{x\to 0^+} lny} = e^0 = 1$$

Example : Evaluate $\lim_{x\to 0^+}(\frac{1}{x})^{sin(x)}$.

Solution : We have an indeterminate form of the type ∞^0. Let

$$y = (\frac{1}{x})^{sin(x)}$$

Then

$$lny = ln(\frac{1}{x})^{sin(x)} = (sin(x))ln(\frac{1}{x})$$

and

$$\lim_{x\to 0^+} lny = \lim_{x\to 0^+} (sin(x))ln\frac{1}{x}$$

This last limit is an indeterminate form of the type $0.\infty$. By writing

$$(sin(x))ln(\frac{1}{x}) = \frac{ln\frac{1}{x}}{\frac{1}{sin(x)}}$$

we can transform it into an indeterminate form of the type ∞/∞ and hence use l'Hopital's Rule. We have

$$\begin{aligned}
\lim_{x\to 0^+} lny &= \lim_{x\to 0^+} \frac{ln\frac{1}{x}}{\frac{1}{sin(x)}} \\
&= \lim_{x\to 0^+} \frac{lnx}{\frac{1}{sin(x)}} \\
&= \lim_{x\to 0^+} \frac{\frac{-1}{x}}{\frac{-cos(x)}{sin^2(x)}} = \lim_{x\to 0^+} \frac{sin^2(x)}{xcos(x)} \\
&= \lim_{x\to 0^+} (\frac{sin(x)}{x})(tan(x)) = 0
\end{aligned}$$

Therefore,

$$\lim_{x\to 0^+} (\frac{1}{x})^{sin(x)} = \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{lny} = e^{\lim_{x\to 0^+} lny} = e^0 = 1$$

Example : Evaluate $\lim_{x\to\infty}(1+\frac{1}{x})^x$.

Solution : We have an indeterminate form of the type 1^∞. Let $y=(1+\frac{1}{x})^x$ Then

$$lny = ln(1+\frac{1}{x})^x = xln(1+\frac{1}{x})$$

so,

$$\lim_{x\to\infty} lny = \lim_{x\to\infty} xln(1+\frac{1}{x})$$

has an indeterminate form of the type $0.\infty$. Rewriting and using l'Hopital's Rule, we obtain

$$\begin{aligned}
\lim_{x\to\infty} lny &= \lim_{x\to\infty} xln(1+\frac{1}{x}) \\
&= \lim_{x\to\infty} \frac{ln(1+\frac{1}{x})}{\frac{1}{x}} \\
&= \lim_{x\to\infty} [\frac{(\frac{1}{1+\frac{1}{x}})(-\frac{1}{x^2})}{\frac{-1}{x^2}}] \\
&= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1
\end{aligned}$$

Therefore,

$$\lim_{x\to\infty}(1+\frac{1}{x})^x = \lim_{x\to\infty} y = \lim_{x\to\infty} e^{lny} = e^{\lim_{x\to\infty} lny} = e^1 = e$$

Module 2

Infinite sequences and series

2.1 Improper Integral

Definition 13. *Integrals with infinite limits of integration are improper integrals of Type I.*

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)dx = \lim_{b \to \infty} \int_a^b f(x)dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_\infty^b (x)dx = \lim_{a \to -\infty} \int_a^b f(x)dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_\infty^\infty f(x)dx = \int_\infty^c f(x)dx + \int_c^\infty f(x)dx,$$

where c is any real number .

In each case, if the limit exists and is finite, we say that the improper integral converges and that the limit is the value of the improper integral. If the limit fails to exist, the improper integral diverges and we say the area under the curve is infinite.

Example : Evaluate $\int_1^\infty \frac{lnx}{x^2}dx$
Solution : Integrating by parts,

$$\begin{aligned}\int_1^b \frac{lnx}{x^2}dx &= [(lnx)(-\frac{1}{x})]_1^b - \int_1^b (-\frac{1}{x})(\frac{1}{x})dx \\ &= -\frac{lnb}{b} - [\frac{1}{x}]_1^b \\ &= -\frac{lnb}{b} - \frac{1}{b} + 1\end{aligned}$$

$$\begin{aligned}\int_1^\infty \frac{lnx}{x^2}dx &= \lim_{b\to\infty}\int_1^b \frac{lnx}{x^2}dx \\ &= \lim_{b\to\infty}[-\frac{lnb}{b} - \frac{1}{b} + 1] \\ &= -[\lim_{b\to\infty}\frac{lnb}{b}] - 0 + 1 \\ &= -[\lim_{b\to\infty}\frac{1/b}{1}] + 1 = 0 + 1 = 1\end{aligned}$$

Example : Evaluate $\int_{-\infty}^\infty \frac{dx}{1+x^2}$.

Solution : $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$

$$\begin{aligned}
\int_{-\infty}^{0} \frac{dx}{1+x^2} &= \lim_{a\to-\infty} \int_{a}^{0} \frac{dx}{1+x^2} \\
&= \lim_{a\to-\infty} tan^{-1}x|_a^0 \\
&= \lim_{a\to-\infty} (tan^{-1}0 - tan^{-1}a) \\
&= \frac{\pi}{2}
\end{aligned}$$

$$\begin{aligned}
\int_{0}^{\infty} \frac{dx}{1+x^2} &= \lim_{b\to\infty} \int_{0}^{b} \frac{dx}{1+x^2} \\
&= \lim_{b\to\infty} tan^{-1}x|_0^b \\
&= \lim_{b\to\infty} (tan^{-1}b - tan^{-1}0) \\
&= \frac{\pi}{2}
\end{aligned}$$

Thus, $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$

Exercise : Examine for what values of p does the integral $\int_{1}^{\infty} \frac{dx}{x^p}$ converge?

Definition 14. *Integrals of functions that become infinite at a point within the interval of integration are improper integrals of Type II.*
1. If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then

$$\int_{a}^{b} f(x)dx = \lim_{c\to a^+} \int_{c}^{b} f(x)dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{c\to b^-} \int_a^c f(x)dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

In each case, if the limit exists and is finite, we say the improper integral converges and that the limit is the value of the improper integral. If the limit does not exist, the integral diverges. In Part 3 of the definition, the integral on the left side of the equation converges if both integrals on the right side converge; otherwise it diverges.

Example : Evaluate $\int_0^1 \frac{dx}{1-x}$.

Solution : Notice that $f(x) = \frac{1}{1-x}$ is continuous at $[0, 1)$ but is discontinuous at $x = 1$.

$$\begin{aligned}\lim_{b\to 1^-} \int_0^b \frac{dx}{1-x} &= \lim_{b\to 1^-} [-ln|1-x|]_0^b \\ &= \lim_{b\to 1^{-1}} [\ln(1-b) + 0] = \infty.\end{aligned}$$

The limit is infinite, so the integral diverges.

2.2 Sequences

Definition 15. *A **sequence** $\{a_n\}$ is a function whose domain is the set of positive integers. The functional values a_1, a_2, a_3, a_n,.... are the **terms** of the sequence, and the term a_n is called the a **nth term** of the sequence.*

Remark: 1. The sequence $\{a_n\}$ is also denoted by $\{a_n\}_{n=1}^{\infty}$.
2. Sometimes it is convenient to begin a sequence with a_k. In this case the sequence is $\{a_n\}_{n=1}^{\infty}$, and its terms are a_k, a_{k+1}, a_{k+2},.., a_n,... .

Example : List the terms of the sequence.
a.$\{\frac{n}{n+1}\}$ **b.**$\{(-1)^n\sqrt{n-2}\}_{n=2}^{\infty}$ **c.**$\{sin\frac{n\pi}{3}\}_{n=0}^{\infty}$
Solution:
a. Here $a_n = f(n) = \frac{n}{n+1}$. Thus,
$a_1 = f(1) = \frac{1}{1+1} = \frac{1}{2}$, $a_2 = f(2) = \frac{2}{2+1} = \frac{2}{3}$, $a_3 = f(3) = \frac{3}{3+1} = \frac{3}{4}$,... and we see that the given sequence can be written as

$$\{\frac{n}{n+1}\} = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, ..., \frac{n}{n+1}, ...\}$$

b. $$\{(-1)^n\sqrt{n-2}\}_{n=2}^{\infty} = \{(-1)^2\sqrt{0}, (-1)^3\sqrt{1}, (-1)^4\sqrt{2}, (-1)^5\sqrt{3}, ..., (-1)^n\sqrt{n-2}, ...\}$$
$$= \{0, -\sqrt{1}, \sqrt{2}, -\sqrt{3}, ..., (-1)^n\sqrt{n-2}, ...\}$$

Note that n starts from 2 in this example. Refer to **Remark: 2**.

$$\textbf{c.}\quad \{sin\frac{n\pi}{3}\}_{n=0}^{\infty} = \left\{ sin0, sin\frac{\pi}{3}, sin\frac{2\pi}{3}, sin\frac{3\pi}{3}, sin\frac{4\pi}{3}, ..., sin\frac{n\pi}{3}, ... \right\}$$
$$= \left\{ 0, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, ..., sin\frac{n\pi}{3}, ... \right\}$$

Note that n starts from 0 in this example. Refer to **Remark: 2**.

Example: Find an expression for the n th term of each sequence.
a. $\{2, \frac{3}{\sqrt{2}}, \frac{4}{\sqrt{3}}, \frac{5}{\sqrt{4}}, ...\}$ **b.** $\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, ...\}$
Solution: a. The terms of the sequence may be written in the form

$$a_1 = \frac{1+1}{\sqrt{1}}, \qquad a_2 = \frac{2+1}{\sqrt{2}}, \qquad a_3 = \frac{3+1}{\sqrt{3}}, \qquad a_4 = \frac{4+1}{\sqrt{4}},$$

Thus, which we see that $a_n = \frac{n+1}{\sqrt{n}}$.
b. Note that $(-1)^r$ is equal to 1 if r is an even integer and -1 if r is an odd integer. Using this result, we obtain

$$a_1 = \frac{(-1)^0}{1}, \qquad a_2 = \frac{(-1)^1}{1}, \qquad a_3 = \frac{(-1)^2}{3}, \qquad a_4 = \frac{(-1)^3}{4},$$

Hence, we can conclude that the nth term is $a_n = \frac{(-1)^{n-1}}{n}$.

Recursively defined sequences: The sequence is defined by specifying the first term or the first few terms of the sequence and a rule for calculating any other term of the sequence from the preceding term(s).

Definition 16. ***Limit of a Sequence***
A sequence a_n has the limit L ,written

$$\lim_{n\to\infty} a_n = L$$

*if a_n can be made as close to L as we please by taking n sufficiently large. If $lim_{n\to\infty}a_n$ exists, we say that the sequence **converges**. Otherwise, we say that the sequence **diverges**.*

We say that a sequence $\{a_n\}$ converges and has the limit L, written

$$\lim_{n\to\infty} a_n = L$$

if for every $\epsilon > 0$ there exists a positive integer N such that $|a_n - L| < \epsilon$ whenever $n > N$.

Theorem 16. *If $\lim_{x\to\infty} f(x) = L$ and $\{a_n\}$ is a sequence defined by $a_n = f(n)$, where n is a positive integer, then $\lim_{n\to\infty} a_n = L$.*

Remark: The converse of the above theorem is false. Consider the sequence $\{sinn\pi\} = \{0\}$. Note that this sequence converges to 0. However, $\lim_{x\to\infty} sin\pi x$ does not exist.

Theorem 17. ***Limit Laws for Sequences***
Suppose that $\lim_{n\to\infty} a_n = L$and $\lim_{n\to\infty} b_n = M$ and that c is a constant. Then

- $\lim_{n\to\infty} ca_n = cL$
- $\lim_{n\to\infty} (a_n \pm b_n) = L \pm M$
- $\lim_{n\to\infty} a_n b_n = LM$

- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$, *provided that* $b_n \neq 0$ *and* $M \neq 0$
- $\lim_{n\to\infty} a_n^p = L^p$, *if* $p > 0$ *and* $a_n > 0$

Theorem 18. ***Squeeze Theorem for Sequences***

If there exists some integer N *such that* $a_n \leq b_n \leq c_n$ *for all* $n \geq N$ *and* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, *then* $\lim_{n\to\infty} b_n = L$.

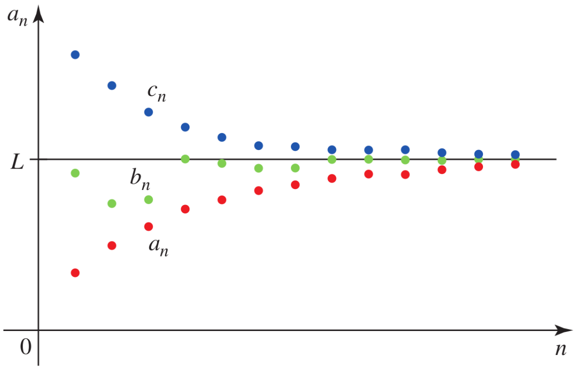


Figure 2.1: The sequence $\{b_n\}$ is squeezed between the sequences $\{a_n\}$ and $\{c_n\}$

.

Example: For $n!$ defined as $n! = n(n-1)(n-2)...1$; find

$$\lim_{n\to\infty} \frac{n!}{n^n}.$$

Solution: Let $a_n = n!/n^n$. Then, the first three terms are given by

$$a_1 = \frac{1!}{1} = 1, \qquad a_2 = \frac{2!}{2} = \frac{2\cdot 1}{2\cdot 2}, \qquad a_3 = \frac{3!}{3} = \frac{3\cdot 2\cdot 1}{3\cdot 3\cdot 3}$$

and the nth term is

$$
\begin{aligned}
a_3 = \frac{n!}{n} &= \frac{n(n-1)\cdot ... \cdot 3 \cdot 2 \cdot 1}{n \cdot n \cdot ... \cdot n \cdot n \cdot n} \\
&= \left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right) \cdot ... \cdot \left(\frac{3}{n}\right)\left(\frac{2}{n}\right)\left(\frac{1}{n}\right) \\
&\leq \left(\frac{1}{n}\right)
\end{aligned}
$$

Therefore,

$$0 < a_n \leq \frac{1}{n}.$$

Since $\lim_{n\to\infty} \frac{1}{n} = 0$, the Squeeze Theorem gives us

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n!}{n^n} = 0$$

Theorem 19. *If* $\lim_{n\to\infty} |a_n| = 0$*, then* $\lim_{n\to\infty} a_n = 0$

Remark: The above theorem is an immediate consequence of the Squeeze Theorem.

Theorem 20. *If* $\lim_{n\to\infty} a_n = L$ *and the function* f *is continuous at* L *, then*

$$\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(L)$$

Definition 17. ***Monotonic Sequences***

A sequence $\{a_n\}$ *is* ***increasing*** *if*

$$a_1 < a_2 < a_3 < ... < a_n < a_{n+1} < ...$$

and ***decreasing*** *if*

$$a_1 > a_2 > a_3 > ... > a_n > a_{n+1} > ...$$

*A sequence is **monotonic** if it is either increasing or decreasing.*

Definition 18. ***Bounded Sequence***
*A sequence $\{a_n\}$ is **bounded above** if there exists a number M such that*

$$a_n \leq M \quad \textit{for all } n \geq 1.$$

*A sequence is **bounded below** if there exists a number m such that*

$$m \leq a_n \quad \textit{for all } n \geq 1.$$

*A sequence is **bounded** if it is both bounded above and bounded below.*

Remarks:

> **a.** A bounded need not be convergent. Note that, the sequence $(-1)^n$ is bounded since $-1 \leq (-1)^n \leq 1$; however, it is evidently divergent.
>
> **b.** A monotonic sequence need not be convergent. Consider the sequence $\{n\}$. This sequence is clearly increasing, yet divergent.

Theorem 21. ***Monotone Convergence Theorem for Sequences***
Every bounded, monotonic sequence is convergent.

Example: Show that $\left\{\frac{2^n}{n!}\right\}$ is convergent and find its limit.
Solution: Here, $a_n = 2^n/n!$. Therefore, the first few terms of the sequence are
$a_1 = 2, a_2 = 2, a_3 \approx 1.333333, a_4 \approx 0.066667, a_5 = 0.266667,$

$a_6 \approx 0.0888889, ...a_{10} \approx 0.000282$

This suggests that the sequence is decreasing from $n = 2$ onward.

To prove this, we compute

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \frac{2^{n+1}n!}{2^n(n+1)!} = \frac{2 \cdot n!}{(n+1)n!} = \frac{2}{n+1}$$

So, we have

$$\frac{a_{n+1}}{a_n} \leq 1 \quad \text{if} \quad n \geq 1.$$

Thus, $a_{n+1} \leq a_n$ if $n \geq 1$ and hence we have proved the assertion. Since all the terms in the sequence are positive, $\{a_n\}$ is bounded below by 0. Therefore, the sequence is decreasing and bounded below, and the Monotone Convergence Theorem for Sequences guarantees that it converges to a non-negative limit L.

Now to find L, consider:

$$a_{n+1} = \frac{2}{n+1}a_n$$

Note that since $\lim_{n\to\infty} a_{n+1} = L$, we also have $\lim_{n\to\infty} a_n = L$. Taking the limit on both sides of the above equation and using Law (3) for limits of sequences, we obtain

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \left(\frac{2}{n+1}a_n\right) = \lim_{n\to\infty} \frac{2}{n+1} \cdot \lim_{n\to\infty} a_n = 0 \cdot L = 0$$

Thus, we can conclude that $\lim_{n\to\infty} 2^n/n! = 0$.

Properties of the Sequence $\{r^n\}$
The sequence $\{r^n\}$ converges if $-1 < r \leq 1$ and

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

It diverges for all other values of r.

2.3 Series

Definition 19. *In general, an expression of the form*

$$a_1 + a_2 + a_3 + ... + a_n + ...$$

is called an ***infinite series*** *or, more simply, a* ***series****. The numbers $a_1, a_2, a_3, ...$ are called the terms of the series; a_n is called the* ***nth term****, or* ***general term****, of the series; and the series itself is denoted by the symbol*

$$\sum_{n=1}^{\infty} a_n$$

or simply $\sum a_n$.

Definition 20. ***Convergence of Infinite Series***
Given an infinite series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ... + a_n + ...$$

the ***nth partial sum*** *of the series is*

$$S_k = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + ... + a_n$$

If the sequence of partial sums $\{S_n\}$ ***converges*** *to the number* S*, that is, if* $\lim_{n\to\infty} S_n = S$*, then the series* $\sum a_n$ ***converges*** *and has* ***sum*** S*, written*

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ... + a_n + ... = S$$

If $\{S_n\}$ *diverges, then the series* $\sum a_n$ ***diverges****.*

Example: Determine whether the series converges. If the series converges, find its sum.

a. $\displaystyle\sum_{n=1}^{\infty} n$ **b.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$

Solution : a. The *n*th partial sum of the series is

$$S_n = 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$$

Since

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(n+1)}{2} = \infty$$

Thus, we can conclude that the limit does not exist and $\sum_{n=1}^{\infty} n$ diverges.

b. The *n*th partial sum of the series is

$$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + ... + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$
$$= 1 - \frac{1}{n+1}$$

Since

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right) = 1$$

we can conclude that

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) = 1$$

Remark: The series in part **b.** of Example is called a **telescoping series**.

Example: Show that the series $\sum_{n=1}^{\infty}\frac{4}{4n^2-1}$ is convergent, and find its sum.

Solution: First, we use partial fraction decomposition to rewrite the general term $a_n = 4/(4n^2-1)$:

$$a_n = \frac{4}{4n^2-1} = \frac{4}{(2n-1)(2n+1)} = \frac{2}{2n-1} - \frac{2}{2n+1}$$

Then we write the *n*th partial sum of the series as

$$\begin{aligned}
S_n &= \sum_{k=1}^{n}\frac{4}{4k^2-1} = \sum_{k=1}^{n}\left(\frac{2}{2k-1}-\frac{2}{2k+1}\right)\\
&= \left(\frac{2}{1}-\frac{2}{3}\right)+\left(\frac{2}{3}-\frac{2}{5}\right)+\left(\frac{2}{5}-\frac{2}{7}\right)+...+\left(\frac{2}{2n-1}-\frac{2}{2n+1}\right)\\
&= 2-\frac{2}{2n+1}
\end{aligned}$$

Note that this is also a telescoping series. Now, since

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(2-\frac{2}{2n+1}\right) = 2$$

we can conclude that the given series is convergent and has sum 2.

2.3.1 Geometric Series

Definition 21. ***Geometric Series***
A series of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ... + ar^{n-1} + ...a \neq 0$$

is called a ***geometric series*** *with common ratio r .*

Theorem 22. *If $|r| < 1$, then the geometric series*

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ... + ar^{n-1} + ...$$

converges, and its sum is $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$. The series diverges if $|r| \geq 1$.

Proof. The *n*th partial sum of the series is

$$S_n = \sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ... + ar^{n-1}$$

Multiplying both sides of this equation by *r* gives

$$rS_n = ar + ar^2 + ar^3 + ... + ar^n$$

Subtracting the second equation from the first then yields $(1 -$

$r)S_n = a(1-r^n)$. If $r \neq 1$, we can solve for S_n, obtaining

$$S_n = \frac{a(1-r^n)}{1-r}$$

Recall that $\lim_{n\to\infty} r^n = 0$ if $|r| < 1$. Then,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}$$

This implies that

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

When $|r| > 1$, the sequence $\{r^n\}$ diverges. So, $\lim_{n\to\infty} S_n$ does not exist. Thus, the geometric series diverges. Verify that $\{S_n\}$ diverges if $r = \pm 1$, implying that the series also diverges for these values of r. $\square$

Example: Express the number $3.2\overline{14} = 3.2141414...$ as a rational number.

Solution: We rewrite the number as

$$\begin{aligned} 3.2141414... &= 3.2 + \frac{14}{10^3} + \frac{14}{10^5} + \frac{14}{10^7} + ... \\ &= \frac{32}{10} + \frac{14}{10^3}\left[1 + \frac{1}{10^2} + \frac{1}{10^4} + ...\right] \\ &= \frac{32}{10} + \sum_{n=1}^{\infty} \left(\frac{14}{10^3}\right)\left(\frac{1}{10^2}\right)^{n-1} \end{aligned}$$

Note that the expression after the first term is a geometric series with $a = \frac{14}{1000}$ and $r = \frac{1}{100}$. Using the theorem above, we have

$$3.2141414... = \frac{32}{10} + \frac{\frac{14}{1000}}{1 - \frac{1}{100}} = \frac{32}{10} + \frac{14}{990} = \frac{3182}{990}$$

2.3.2 The Harmonic Series

Definition 22. *The series*

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2 + \frac{1}{3}} + ...$$

is called the ***harmonic series***.

For showing that this series is divergent, firstly note that: If a sequence $\{b_n\}$ is convergent, then any *subsequence* obtained by deleting any number of terms from the parent sequence $\{b_n\}$ must also converge to the same limit. Therefore, to show that a sequence is divergent, it suffices to produce a subsequence of the parent sequence that is divergent.
Thus, it would be sufficient to show that the subsequence
$S_2, S_4, S_8, S_{16}, ..., S_{2^n}, ...$
of the sequence $\{S_n\}$ of partial sums of the harmonic series is divergent.

$$\begin{aligned}
S_2 &= 1 + \frac{1}{2} \\
S_4 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + 2\left(\frac{1}{2}\right) \\
S_8 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + 3\left(\frac{1}{2}\right) \\
S_{16} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + ... + \frac{1}{8}\right) + \left(\frac{1}{9} + ... + \frac{1}{16}\right) \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + ... + \frac{1}{8}\right) + \left(\frac{1}{16} + ... + \frac{1}{16}\right) \\
&= 1 + 4\left(\frac{1}{2}\right)
\end{aligned}$$

Thus, $S_{2^n} > 1 + n\left(\frac{1}{2}\right)$. Therefore, $\lim\limits_{n\to\infty} S_{2^n} = \infty$. So, $\{S_n\}$ is divergent. This proves that the harmonic series is divergent.

2.3.3 The Divergence Test

Theorem 23. *If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim\limits_{n\to\infty} a_n = 0$.*

Proof. We can write the partial sum as $S_n = a_1 + a_2 + ... + a_{n-1} + a_n = S_{n-1} + a_n$, so $a_n = S_n - S_{n-1}$. Since $\sum_{n=1}^{\infty} a_n$ is convergent, the sequence $\{S_n\}$ is convergent. Let $\lim\limits_{n\to\infty} S_n = S$. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0$$

$\square$

Theorem 24. ***The Divergence Test***

If $\lim\limits_{n\to\infty} a_n$ *does not exist or* $\lim\limits_{n\to\infty} a_n \neq 0$*, then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Remark: The Divergence Test does not say that if $\lim\limits_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ must converge. In other words, the converse of 2.3.3is not true in general.

Example: Show that the following series are divergent.

a. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}$ **b.** $\displaystyle\sum_{n=1}^{\infty}\frac{2n^2+1}{3n^2-1}$

Solution: **a.** Observe that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}(-1)^{n-1}$ does not exist. Thus, by divergence test the series diverges.

b.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{2n^2+1}{3n^2-1} = \lim_{n\to\infty}\frac{2+\frac{1}{n^2}}{3-\frac{1}{n^2}} = \frac{2}{3} \neq 0$$

Thus, by the Divergence Test, the series diverges.

2.3.4 Properties of Convergent Series

Theorem 25. ***Properties of Convergent Series***

If $\sum_{n=1}^{\infty} a_n = A$ *and* $\sum_{n=1}^{\infty} b_n = B$ *are convergent and* c *is any real number, then* $\sum_{n=1}^{\infty} ca_n$ *and* $\sum_{n=1}^{\infty}(a_n \pm b_n)$ *are also convergent, and*

a. $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n = cA$

b. $\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n = A \pm B$

Example: Show that the series $\sum_{n=1}^{\infty}\left[\frac{2}{n(n+1)} - \frac{4}{3^n}\right]$ is convergent, and find its sum.

Solution: Consider $\sum_{n=1}^{\infty} 1/[n(n+1)]$. Using partial fraction decomposition, and previous example,

$$1 = \sum_{n=1}^{\infty} \frac{1}{[n(n+1)]} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

Observe that $\sum_{n=1}^{\infty} \frac{4}{3^n}$ is a geometric series with $a = \frac{4}{3}$ and $r = \frac{1}{3}$. Thus, $\sum_{n=1}^{\infty} \frac{4}{3^n} = \frac{\frac{4}{3}}{1-\frac{1}{3}} = 2$. Thus, by **Properties of Convergent Series** the given series is convergent and

$$\sum_{n=1}^{\infty} \left[\frac{2}{n(n+1)} - \frac{4}{3^n}\right] = 2\sum_{n=1}^{\infty} \frac{1}{n(n+1)} - \sum_{n=1}^{\infty} \frac{4}{3^n} = 2 \cdot 1 - 2 = 0$$

2.4 The Integral Test

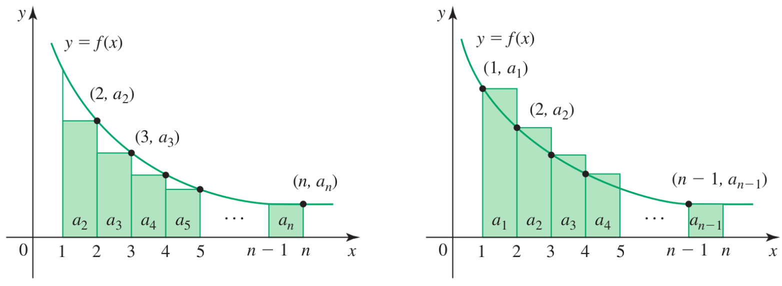
Theorem 26. ***The Integral Test*** *Suppose that f is a continuous, positive, and decreasing function on $[1, \infty)$ if $f(n) = a_n$ for $n \geq 1$, then*

$$\sum_{n=1}^{\infty} a_n \qquad \textit{and} \qquad \int_1^{\infty} f(x)dx$$

either both converge or both diverge.

Proof. In **Figure (a)** given, observe that the height of the first rectangle is $a_2 = f(2)$. Since this rectangle has width 1, the area of the rectangle is also $a_2 = f(2)$.Similarly, the area of the second rectangle is a_3, and so on. Comparing the sum of the areas of the first $(n-1)$ inscribed rectangles with the area of the region under the graph of f over the interval $[1, n]$, we see that

$$a_2 + a_3 + ... + a_n \leq \int_1^n f(x)dx$$



(a) $a_2 + a_3 + \cdots + a_n \leq \int_1^n f(x)\,dx$ **(b)** $\int_1^n f(x)\,dx \leq a_1 + a_2 + \cdots + a_{n-1}$

which gives us the following result. The second inequality follows if we consider $\int_1^n f(x)dx$ is convergent and has value L.

$$S_n = a_1 + a_2 + a_3 + ... + a_n \leq a_1 + \int_1^n f(x)dx \leq a_1 + L$$

This shows that $\{S_n\}$ is bounded above. Also, since $a_{n+1} = f(n+1) \geq 0$, we have that $S_{n+1} = S_n + a_{n+1} \geq S_n$ which proves that $\{S_n\}$ is increasing as well. Therefore, by Monotone Convergence Theorem for Sequences, $\{S_n\}$ is convergent. Consequently, $\sum_{n=1}^{\infty} a_n$ is convergent.
Next, by examining **Figure(b)** we can see that

$$\int_1^n f(x)dx \leq a_1 + a_2 + a_3 + ... + a_{n-1} = S_{n-1}$$

Thus, if $\int_1^n f(x)dx$ diverges (to ∞ since $f(x) \geq 0$),then $\lim\limits_{n\to\infty} S_{n-1} = \lim\limits_{n\to\infty} S_n = \infty$, and $\sum_{n=1}^{\infty} a_n$ is divergent. $\square$

Example: Use the Integral Test to determine whether the following converges or diverges.

$$\textbf{a.} \sum_{n=1}^{\infty} \frac{1}{n^2+1} \qquad \textbf{b.} \sum_{n=1}^{\infty} \frac{ln\, n}{n}$$

Solution: a. Here $a_n = f(n) = 1/(n^2+1)$, giving $f(x) = 1/(x^2+1)$. Since f is continuous, positive, and decreasing on $[1, \infty)$, we may use the Integral Test.

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^2+1} dx &= \lim_{b\to\infty} \int_1^b \frac{1}{x^2+1} dx \\
&= \lim_{b\to\infty} [tan^{-1} x]_1^b \\
&= \lim_{b\to\infty} (tan^{-1} b - tan^{-1} 1) \\
&= \frac{\pi}{2} - \frac{\pi}{4} \\
&= \frac{\pi}{4}
\end{aligned}$$

Since $\int_1^{\infty} \frac{1}{x^2+1} dx$ converges, so does $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.
b. Here $a_n = (ln\, n)/n$. Thus, $f(x) = (ln\, x)/x$. Observe that f is continuous and positive on $[1, \infty)$. Now to see if it is decreasing or not. Consider

$$f'(x) = \frac{x\left(\frac{1}{x}\right) - ln\, x}{x^2} = \frac{1 - ln\, x}{x^2}$$

Note that $f' < 0$ when $ln\, x > 1$, i.e., if $x > e$. Hence, f is decreasing on $[3, \infty)$. Therefore, we can use the Integral Test:

$$\begin{aligned}\int_3^\infty \frac{ln\ x}{x} dx &= \lim_{b\to\infty} \int_3^b \frac{ln\ x}{x} dx \\ &= \lim_{b\to\infty} \left[\frac{1}{2}(ln\ x)^2\right]_3^b \\ &= \lim_{b\to\infty} \frac{1}{2}[(ln\ b)^2 - (ln\ 3)^2] \\ &= \infty\end{aligned}$$

We can conclude that $\sum_{n=1}^{\infty} \frac{ln\ n}{n}$ diverges.

The *p* -Series

Definition 23. ***The p -Series***
*A **p -series** is a series of the form*

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + ...$$

where p is a constant.

Remark: Note that when $p = 1$, the *p* -series is just the harmonic series.

Theorem 27. ***Convergence of the p -Series***
The p -series a p converges if $p > 1$ *and diverges if* $p \leq 1$.

Proof. When $p < 0$, $\lim_{n\to\infty} \frac{1}{n^p} = \infty$ and if $p = 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 1$. In either case, $\lim_{n\to\infty} \frac{1}{n^p} \neq 0$, so the *p*-series diverges by the Divergence Test. Recall that $\int_1^\infty 1/x^p dx$ converges if $p > 1$ and diverges if $p \leq 1$. Using this and by the Integral Test, we have that $\sum_{n=1}^{\infty} 1/n^p$ converges if $p > 1$ and diverges if $p \leq 1$. $\square$

Example: Determine whether the given series converges or diverges.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ **b.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ **c.** $\displaystyle\sum_{n=1}^{\infty} n^{-1.001}$

Solution: **a.** This is a *p*-series with $p = 2 > 1$. Thus it converges by theorem.
b. This is a *p*-series with $p = 1/2 < 1$. Thus it diverges by theorem.
c. This is a *p*-series with $p = 1.001 > 1$. Thus it converges by theorem.

2.5 The Comparison Tests

2.5.1 The Comparison Test

Theorem 28. *Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.*
a. *If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.*
b. *If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.*

Proof. Let

$$S_n = \sum_{k=1}^{n} a_k \quad \text{and} \quad T_n = \sum_{k=1}^{n} b_k$$

be the nth terms of the sequence of partial sums of $\sum a_n$ and $\sum b_n$, respectively. Since both series have positive terms, $\{S_n\}$ and $\{T_n\}$ are increasing.

a. If $\sum_{n=1}^{\infty} b_n$ is convergent, then there exists a number L such that $\lim_{n\to\infty} T_n = L$ and $T_n \leq L$ for all n. Since $a_n \leq b_n$ for all n, we have $S_n \leq T_n$, and this implies that $S_n \leq L$ for all n. We have shown that $\{S_n\}$ is increasing and bounded above, so by the Monotone Convergence Theorem for Sequences, $\sum a_n$ converges.

b. If $\sum_{n=1}^{\infty} b_n$ is divergent, then $\lim_{n\to\infty} T_n = \infty$ since $\{T_n\}$ increasing. But $a_n \geq b_n$ for all n, we have $S_n \geq T_n$, and this implies that $\lim_{n\to\infty} S_n = \infty$. Therefore, $\sum a_n$ diverges. $\square$

Example: Determine whether the following series are convergent or divergent.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3+2^n}$ **b.** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$

Solution: a. Let $a_n = \frac{1}{3+2^n}$. For large n, $3 + 2^n$ behaves like 2^n. So, a_n behaves like $b_n = (\frac{1}{2}^n)$. Comparing $\sum a_n$ with $\sum b_n$, we see that $\sum b_n$ is a geometric series with $r = \frac{1}{2} < 1$, implying that it is convergent. Observe that

$$a_n = \frac{1}{3+2^n} < \frac{1}{2^n} = b_n \qquad n \geq 1$$

Thus by Comparison test, the given series is convergent.

Remark: Since the convergence or divergence of a series is not affected by the omission of a finite number of terms of the series, the condition $a_n \geq b_n$ (or $a_n \leq b_n$) for all n can be replaced by the condition that these inequalities hold for all $n \geq N$ for some integer N.

b. Let $a_n = \frac{1}{\sqrt{n}-1}$. For n large, $\sqrt{n} - 1$ behaves like $\sqrt{n}$, so a_n behaves like $b_n = \frac{1}{\sqrt{n}}$. Note that the series $\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \sqrt{n}$ is a *p*-series with $p = \frac{1}{2}, 1$, thus making it divergent. Since

$$a_n = \frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}} = b_n \qquad \text{for} \qquad n \geq 2$$

we get that the given series id divergent by Comparison test.

2.5.2 The Limit Comparison Test

Theorem 29. ***The Limit Comparison Test***
Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

where L is a positive number. Then either both series converge or both diverge.

Proof. Since $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = L > 0$, there exists an integer N such that $n \geq N$ implies that

$$\left|\frac{a_n}{b_n} - L\right| < \frac{1}{2}L \implies \frac{1}{2}L < \frac{a_n}{b_n} < \frac{3}{2}L \quad \text{OR} \quad \frac{1}{2}Lb_n < a_n < \frac{3}{2}Lb_n$$

If $\sum b_n$ converges, so does $\sum \frac{3}{2}Lb_n$. Therefore, the right side of the last inequality implies that $\sum a_n$ converges by the Comparison Test. On the other hand, if $\sum b_n$ diverges, so does $\sum \frac{1}{2}Lb_n$, and the left side of the last inequality implies by the Comparison Test that $\sum a_n$ diverges as well. $\square$

Example: Determine whether the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}+ln\ n}{n^2+1}$ converges or diverges.

Solution: Note that n large, $\sqrt{n} + ln\ n$ behaves like $\sqrt{n}$. We can verify this by computing the derivatives of $f(x) = \sqrt{x}$ and $g(x) = ln\, x$:

$$f'()x = \frac{1}{2\sqrt{x}} \qquad \text{and} \qquad g'(x) = \frac{1}{x}$$

Observe that $g'(x)$ approaches zero faster than $f'(x)$ approaches zero, as $x \longrightarrow \infty$. This shows that $\sqrt{x}$ grows faster than $ln\ x$. Also, if n is large, $n^2 + 1$ behaves like n^2. Therefore,

$$a_n = \frac{\sqrt{n} + ln\ n}{n^2 + 1} \qquad \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}} = b_n$$

Next we compute,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{1/2} + ln\ n}{n^2+1} \cdot \frac{n^{3/2}}{1} = \lim_{n\to\infty} \frac{n^2 + n^{3/2} ln\ n}{n^2+1}$$
$$= \lim_{n\to\infty} \frac{1 + \frac{ln\ n}{n^{1/2}}}{1 + \frac{1}{n^2}}$$

The last equality is obtained by dividing the numerator and denominator by n^2. Now, note that by l'Hôpital's Rule,

$$\lim_{x\to\infty} \frac{ln\ n}{n^{1/2}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$$

This result also supports the observation made earlier that $\sqrt{x}$ grows faster than $ln\ x$. Using this, we see that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1 + \frac{ln\ n}{n^{1/2}}}{1 + \frac{1}{n^2}} = 1$$

Since $\sum 1/n^{3/2}$ converges (as it is a p-series with $p = \frac{3}{2}$), the given series converges, by the Limit Comparison Test.

2.6 Alternating Series

Theorem 30. ***The Alternating Series Test***
If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + ... \qquad a_n > 0$$

satisfies the conditions
1. $a_{n+1} \leq a_n$ *for all* n
2. $\lim_{n\to\infty} a_n = 0$
then the series converges.

Proof. Consider the subsequence $\{S_{2n}\}$ comprising the even terms

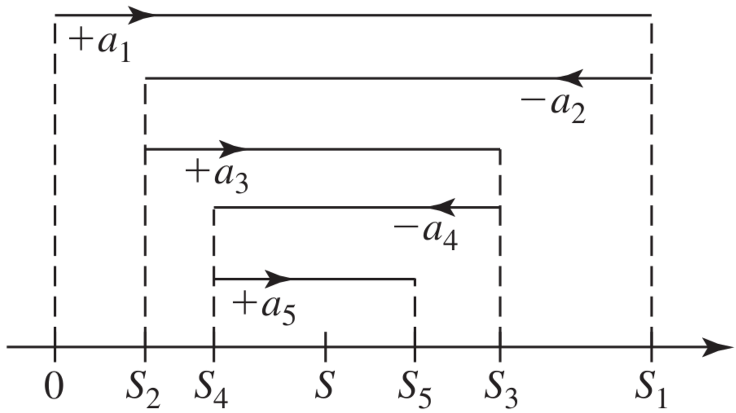


Figure 2.2: Note that the terms of $\{S_n\}$ oscillate in smaller and smaller steps, and this suggests that $\lim\limits_{n\to\infty} S_n = S$.

of the sequence of partial sums $\{S_n\}$. Now,

$$S_2 = a_1 - a_2 \geq 0 \qquad \text{Since } a_1 \geq a_2$$

$$S_4 = S_2 + (a_3 - a_4) \geq S_2 \qquad \text{Since } a_3 \geq a_4$$

.

.

$$S_{2n+2} = S_{2n} + (a_{2n+1} - a_{2n+2}) \geq S_{2n} \qquad \text{Since } a_{2n+1} \geq a_{2n+2}$$

Thus, we have that $0 \leq S_2 \leq S_4 \leq ... \leq S_{2n} \leq ....$ That is, $\{S_{2n}\}$ is increasing. Note that we can write S_{2n} as

$$S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - ... - (a_{2n-2} - a_{2n-1}) - a_{2n}$$

where every expression within the parenthesis is nonnegative. Thus, $S_{2n} \leq a_1$ for all n. This shows that the sequence $\{S_{2n}\}$ is bounded

above as well. Therefore, by the Monotone Convergence Theorem for Sequences, the sequence $\{S_{2n}\}$ is convergent; that is, there exists a number S such that $\lim_{n\to\infty} S_{2n} = S$.
Now, consider the subsequence $\{S_{2n+1}\}$ comprising the even terms of $\{S_n\}$. Since $S_{2n+1} = S_{2n} + a_{2n+1}$ and $\lim_{n\to\infty} a_{2n+1} = 0$ by assumption, we have

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} (S_{2n+1} + a_{2n+1}) = \lim_{n\to\infty} S_{2n+1} + \lim_{n\to\infty} a_{2n+1} = S$$

Since the subsequences $\{S_{2n}\}$ and $\{S_{2n+1}\}$ of the sequence of partial sums $\{S_n\}$ both converge to S, we have $\lim_{n\to\infty} S_n = S$, so the series converges. $\square$

Example : Determine whether the series converges or diverges.

a. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{2n}{4n-1}$ **b.** $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{3n}{4n^2-1}$

Solution: **a.** Here, $a_n = 2n/(4n-1)$. Now,

$$\lim_{n\to\infty} \frac{2n}{4n-1} = \frac{1}{2} \neq 0$$

Observe that condition (2) of Alternating Series Test is not satisfied. In fact, note that the $\lim_{n\to\infty} (-1)^n \frac{2n}{4n-1}$ does not exist, and the divergence of the series follows from the Divergence Test.
b. Here $a_n = 3n/(4n^2-1)$. Consider

$$f'(x) = \frac{(4x^2-1)(3) - (3x)(8x)}{(4x^2-1)^2} = \frac{-12x^2-3}{(4x^2-1)^2}$$

Thus, condition (1) that is, $a_n \geq a_{n+1}$ for all n for the Alternating Series test is verified by showing that $f(x) = 3x/(4x^2-1)$ is decreasing for $x \geq 0$. Now, for condition (2).

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n}{4n^2-1} = \lim_{n\to\infty} \frac{\frac{3}{n}}{4-\frac{1}{n^2}} = 0$$

Therefore, both conditions of the Alternating Series Test are satisfied. We can conclude that the series is convergent.

Approximating the Sum of an Alternating Series by $\mathbf{S_n}$

The sum of a convergent series can be approximated to any degree of accuracy by its nth partial sum S_n, provided that n is taken large enough. To measure the accuracy of the approximation, consider the quantity

$$R_n = S - S_n = \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{n} a_k = \sum_{k=n+1}^{\infty} a_k = a_{n+1} + a_{n+2} + a_{n+3} + \ldots$$

called the **remainder after n terms** of the series $\sum_{n=1}^{\infty} a_k$. The remainder measures the error incurred when S is approximated by S_n.

Theorem 31. ***Error Estimate in Approximating an Alternating Series***

Suppose $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ *is an alternating series satisfying*

1. $0 \leq a_{n+1} \leq a_n$ *for all* n

2. $\lim_{n\to\infty} a_n = 0$

If S is the sum of the series, then

$$|R_n| = |S - S_n| \leq a_{n+1}$$

In other words, the absolute value of the error incurred in approximating S by S_n is no larger than a_{n+1}, the first term omitted.

Proof. We have

$$\begin{aligned} S - S_n &= \sum_{k=1}^{\infty}(-1)^{k-1}a_k - \sum_{k=1}^{n}(-1)^{k-1}a_k = \sum_{k=n+1}^{\infty}(-1)^{k-1}a_k \\ &= (-1)^n a_{n+1} + (-1)^{n+1}a_{n+2} + (-1)^{n+2}a_{n+3} + ... \\ &= (-1)^n(a_{n+1} - a_{n+2} + a_{n+3} - ...) \end{aligned}$$

Note that, since $a_{n+1} \leq a_n$ for all n,

$$a_{n+1} - a_n \geq 0 \qquad \text{for all } n$$

and thus we have

$$\begin{aligned} |S - S_n| &= a_{n+1} - a_{n+2} + a_{n+3} - a_{n+4} + a_{n+5} - ... \\ &= a_{n+1} - (a_{n+2} - a_{n+3}) - (a_{n+4} - a_{n+5}) - ... \\ &\leq a_{n+1} \end{aligned}$$

□

Example: Show that the series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n!}$ is convergent, and find its sum correct to three decimal places.

Solution: Note that for all n,

$$a_{n+1} = \frac{1}{(n+1)!} = \frac{1}{n!(n+1)} < \frac{1}{n!} = a_n \qquad \text{and}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n!} = 0$$

Thus, we conclude that the series converges by the Alternating Series Test. Now to compute the sum consider the reminder term

$$|R^n| = |S - S_n| \leq a_{n+1} = \frac{1}{(n+1)!}$$

We need $|R_n| < 0.0005$, which is satisfied if $\frac{1}{(n+1)!} < 0.0005$ or $(n+1)! > \frac{1}{0.0005} = 2000$. The smallest positive integer that satisfies the last inequality is $n = 6$. Hence, the required approximation is

$$S \approx S_6 = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} \approx 0.368$$

2.7 Absolute Convergence; the Ratio and Root Tests

2.7.1 Absolute Convergence

Definition 24. ***Absolutely Convergent Series***
A series $\sum a_n$ is ***absolutely convergent*** *if the series $\sum |a_n|$ is convergent.*

Example: Show that the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$$

is not absolutely convergent.

Solution: Consider

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...$$

which is the divergent harmonic series. This shows that the series is not absolutely convergent.

Definition 25. ***Conditionally Convergent Series***
*A series $\sum a_n$ is **conditionally convergent** if it is convergent but not absolutely convergent.*

Theorem 32. *If a series $\sum a_n$ is absolutely convergent, then it is convergent.*

Proof. Using an absolute value property, we have $-|a_n| \leq a_n \leq |a_n|$ and now adding $|a_n|$ to both sides gives $0 \leq a_n + |a_n| \leq 2|a_n|$. Let $b_n = a_n + |a_n|$. If $\sum a_n$ is absolutely convergent, then $\sum |a_n|$ is convergent, which in turn implies, by Theorem part**(a)** of Properties of Convergent Series that $\sum 2|a_n|$ is convergent. Therefore, $\sum b_n$ is convergent by the Comparison Test. Finally, since $a_n = b_n - |a_n|$, we see that $\sum a_n = \sum b_n - \sum |a_n|$ is convergent by Theorem part**(b)** of Properties of Convergent Series. $\square$

2.7.2 Ratio Test

Theorem 33. ***The Ratio Test***
Let $\sum a_n$ be a series with nonzero terms.
a. *If $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.*
b. *If $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$, or $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.*
c. *If $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the test is inconclusive, and another test should be used.*

Proof. Suppose that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$$

Let r be any number such that $0 \le L < r < 1$. Then there exists an integer N such that

$$\left|\frac{a_{n+1}}{a_n}\right| < r \qquad \text{OR} \qquad |a_{n+1}| < |a_n| r \qquad \text{for } n \ge N$$

Letting n take on the values $N, N+1, N+2$, ..., successively, we obtain

$$\begin{aligned}
|a_{N+1}| &< |a_N| r\\
|a_{N+2}| &< |a_{N+1}| r < |a_N| r^2\\
|a_{N+3}| &< |a_{N+2}| r < |a_N| r^3\\
&.\\
&.
\end{aligned}$$

$$\text{In general, } |a_{N+k}| < |a_N| r^k \qquad \text{for all} \qquad k \ge 1$$

Observe that the series $\sum_{k=1}^{\infty} |a_N| r^k$ (series(1)) is a convergent geometric series with $0 < r < 1$ and each term of the series $\sum_{k=1}^{\infty} |a_{N+k}|$ (series(2)) is less than the corresponding term of the geometric series (1). The Comparison Test then implies that series (2) is convergent. Since convergence or divergence is unaffected by the omission of a finite number of terms, we see that the series $\sum_{n=1}^{\infty} |a_n|$ is also convergent.

b. Suppose that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$$

Let r be any number such that $L > r > 1$. Then there exists an integer N such that

$$\left| \frac{a_{n+1}}{a_n} \right| > r > 1 \qquad \text{whenever} \qquad n \geq N$$

This implies that $|a_{n+1}| < |a_n|$ when $n \geq N$. Thus, $\lim\limits_{n\to\infty} a_n \neq 0$, and $\sum a_n$ is divergent by the Divergence Test.

c. Consider the series $\sum_{k=1}^{\infty} 1/n$ and $\sum_{k=1}^{\infty} 1/n^2$. For the first series we have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$$

and for the second series,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1}{(n+1)^2} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^2} = 1$$

The first series is the divergent harmonic series, whereas the second series is a convergent *p*-series with $p = 2$. Thus, if $L = 1$, the series may converge or diverge, and the Ratio Test is inconclusive. $\square$

Example: Determine whether the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ is convergent or divergent.

Solution: Let $a_n = n!/n^n$. Then,

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}\\
&= \lim_{n\to\infty}\frac{(n+1)n!}{(n+1)(n+1)^n}\cdot\frac{n^n}{n!}\\
&= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}\\
&= \frac{1}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1
\end{aligned}$$

Therefore, the series converges, by the Ratio Test.

2.7.3 The Root Test

Theorem 34. ***The Root Test***

Let $\sum_{n=1}^{\infty} a_n$ be a series

a. *If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.*

b. *If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, or $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \infty$ then $\sum_{n=1}^{\infty} a_n$ diverges.*

c. *If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 1$,the test is inconclusive, and another test should be used.*

Example: Determine whether the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^{n+3}}{(n+1)^n}$ is absolutely convergent, conditionally convergent, or divergent.

Solution: We apply the Root Test with $a_n = (-1)^{n-1}2^{n+3}/(n+1)^n$. We have

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{(-1)^{n-1}\frac{2^{n+3}}{(n+1)^n}} = \lim_{n\to\infty} \left|\frac{2^{n+3}}{(n+1)^n}\right|^{1/n}$$
$$= \lim_{n\to\infty} \left|\frac{2^{1+3/n}}{n+1}\right| = 0 < 1$$

Thus, the series is absolutely convergent.

2.7.4 Rearrangement of Series

Example: Consider the alternating harmonic series that converges to ln 2

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + ... = ln2$$

If we rearrange the series so that every positive term is followed by two negative terms, we obtain

$$\begin{aligned}
1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + ...& \\
= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + ...& \\
= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + ...& \\
= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + ...\right)& \\
= \frac{1}{2}ln2&
\end{aligned}$$

Thus, rearrangement of the alternating harmonic series has a sum that is one half that of the original series!

NOTE:

- Reimann proved that:
  If x is any real number and $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, then there is a rearrangement of $\sum_{n=1}^{\infty} a_n$ that converges to x.

  Remark: Riemann's result tells us that for conditionally convergent series, we may not rearrange their terms, lest we end up with a totally different series, that is, a series with a different sum. In fact, for conditionally convergent series, one can find rearrangements of the series that diverge to infinity, diverge to minus infinity, or oscillate between any two prescribed real numbers!

- **Q: What kind of convergent series will have rearrangements that converge to the same sum as the original series?**

  If $\sum_{n=1}^{\infty} a_n$ converges absolutely and $\sum_{n=1}^{\infty} b_n$ is any rearrangement of $\sum_{n=1}^{\infty} a_n$, then $\sum_{n=1}^{\infty} b_n$ converges and $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n$.

- Since a convergent series with positive terms is absolutely convergent, its terms can be written in any order, and the resultant series will converge and have the same sum as the original series.

Example: Indicate the test(s) that you would use to determine whether the series converges or diverges.

a. $\displaystyle\sum_{n=1}^{\infty}\frac{2n-1}{3n+1}$ **b.** $\displaystyle\sum_{n=1}^{\infty}\left[\frac{2}{3^n}-\frac{1}{n(n+1)}\right]$ **c.** $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^e$

d. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n\sqrt{\ln n}}$ **e.** $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$ **f.** $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n^3+2}}{n^4+3n^2+1}$

g. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{n^2+1}$ **h.** $\displaystyle\sum_{n=1}^{\infty}\frac{n}{2^n}$ **i.** $\displaystyle\sum_{n=1}^{\infty}\frac{\sin n}{\sqrt{n^3+1}}$

Solution:

a. Since

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{2n-1}{3n-1}=\frac{2}{3}\neq 0$$

we use the Divergence Test.

b. The series is the difference of a geometric series and a telescoping series, so we use the properties of these series to determine convergence.

c. Here, $a_n=\left(\frac{1}{n}\right)^e=\frac{1}{n^e}$ is a p-series, so we use the properties of a p-series to study its convergence.

d.The function $f(x)=\frac{1}{x\sqrt{\ln x}}$ is continuous, positive, and decreasing on $[3,\infty)$ and is integrable, so we choose the Integral Test.

e. Here,

$$a_n=\frac{\ln n}{n^2}<\frac{\sqrt{n}}{n^2}=\frac{1}{n^{3/2}}=b_n$$

and we use the Comparison Test with the test series $\sum b_n$.

f. $a_n=\frac{(n^3+2)^{1/2}}{n^4+3n^2+1}$ is positive and behaves like $b_n=\frac{(n^3)^{1/2}}{n^4}=\frac{n^{3/2}}{n^4}=\frac{1}{n^{5/2}}$ for large values of n, so we use the Limit Comparison

Test with test series $\sum_{n=1}^{\infty} 1/n^{5/2}$.
g. This is an alternating series, and we use the Alternating Series Test.
h. Here, $a_n = \frac{n}{2^n} = \left(\frac{n^{1/n}}{2}\right)^n$ involves the nth power, so the Root Test is a candidate.In fact, here $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \frac{\sqrt[n]{n}}{2} = \frac{1}{2} < 1$ and the series converges.
i. The series involves both positive and negative terms and is not an alternating series, so we use the test for absolute convergence.

Summary of the Convergence and Divergence Tests for Series

1. The **Divergence Test** often settles the question of convergence or divergence of a series $\Sigma\, a_n$ simply and quickly:

   If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges.

2. If you recognize that the series is
   a. a **geometric series** $\Sigma_{n=1}^{\infty} ar^{n-1}$, then it converges with sum $a/(1-r)$ if $|r| < 1$. If $|r| \geq 1$, the series diverges.
   b. a **telescoping series,** then use partial fraction decomposition (if necessary) to find its nth partial sum S_n. Next determine convergence or divergence by evaluating $\lim_{n\to\infty} S_n$.
   c. a ***p*-series** $\Sigma_{n=1}^{\infty} 1/n^p$, then the series converges if $p > 1$ and diverges if $p \leq 1$.

   Sometimes a little algebraic manipulation might be required to cast the series into one of these forms. Also, a series might involve a combination (for example, a sum or difference) of these series.

3. If $f(n) = a_n$ for $n \geq 1$, where f is a continuous, positive, decreasing function on $[1, \infty)$ and readily integrable, then we may use the **Integral Test:**

   $\Sigma_{n=1}^{\infty} a_n$ converges if $\int_1^{\infty} f(x)\, dx$ converges and diverges if $\int_1^{\infty} f(x)\, dx$ diverges.

4. If a_n is positive and behaves like the nth term of a geometric or p-series for large values of n, then the Comparison Test or Limit Comparison Test may be used. The tests and conclusions follow:
   a. If $a_n \leq b_n$ for all n and $\Sigma\, b_n$ converges, then $\Sigma\, a_n$ converges.
   b. If $a_n \geq b_n \geq 0$ for all n and $\Sigma\, b_n \geq 0$ diverges, then $\Sigma\, a_n$ diverges.
   c. If b_n is positive and $\lim_{n\to\infty}(a_n/b_n) = L > 0$, then both series converge or both diverge.

   The comparison tests can also be used on $\Sigma\, |a_n|$ to test for absolute convergence.

5. If the series is an **alternating series,** $\Sigma_{n=1}^{\infty}(-1)^n\, a_n$ or $\Sigma_{n=1}^{\infty}(-1)^{n-1}\, a_n$, then the Alternating Series Test should be considered:

   If $a_n \geq a_{n+1}$ for all n and $\lim_{n\to\infty} a_n = 0$, then the series converges.

6. The **Ratio Test** is useful if a_n involves factorials or nth powers. The series
   a. converges absolutely if $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| < 1$.
   b. diverges if $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| > 1$ or $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \infty$.

   The test is inconclusive if $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 1$.

7. The **Root Test** is useful if a_n involves nth powers. The series
   a. converges absolutely if $\lim_{n\to\infty}\sqrt[n]{|a_n|} < 1$.
   b. diverges if $\lim_{n\to\infty}\sqrt[n]{|a_n|} > 1$ or $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \infty$.

   The test is inconclusive if $\lim_{n\to\infty}\sqrt[n]{|a_n|} = 1$.

8. If the series $\Sigma\, a_n$ involves terms that are both positive and negative but it is not alternating, then one sometimes can prove convergence of the series by proving that $\Sigma\, |a_n|$ is convergent.

Module 3

Power series, plane curves and polar coordinates

3.1 Power series

Definition 26. *Let x be a variable. A power series in x is a series of the form*

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + ...$$

where the a_n's are constants and are called the coefficients of the series. More generally, a power series in $(x - c)$, where c is a constant, is a series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + ... + a_n(x-c)^n + ...$$

Notes :

1. A power series in $(x-c)$ is also called a power series centered at c or a power series about c. Thus, a power series in x is just a series centered at the origin.
2. To simplify the notation used for a power series, we have adopted the convention that $(x-c)^0 = 1$, even when $x = c$.

We can view a power series as a function f defined by the rule

$$f(x) = \sum_{n=0}^{\infty} = a_n(x-c)^n$$

The domain of f is the set of all x for which the power series converges, and the range of f comprises the sums of the series obtained by allowing x to take on all values in the domain of f. If a function f is defined in this manner, we say that f is represented by the power series $f(x) = \sum_{n=0}^{\infty} = a_n(x-c)^n$.

Example : Consider the power series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + ... + x^n + ...$$

Notice that this is a geometric series with common ratio x, we see that it converges for $-1 \leq x \leq 1$. Thus, the power series is a rule for a function f with interval $(-1, 1)$ as its domain; that is,

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + ... + x^n + ...$$

There is a simple formula for the sum of the geometric series, namely, $1/(1-x)$. and we see that the function represented by the series is the function

$$f(x) = \frac{1}{1-x} \quad -1 \leq x \leq 1$$

Even though the domain of the function $g(x) = 1/(1-x)$ is the set of all real numbers except $x = 1$, the power series represents the function $g(x) = 1/(1-x)$ only in the interval of convergence $(-1, 1)$ of the series.

Theorem 35. ***(Convergence of a power series)*** *Given a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$, exactly one of the following is true:*
a. The series converges only at $x = c$.
b. The series converges for all x.
c. There is a number $R > 0$ such that the series converges for $|x-c| < R$ and diverges for $|x-c| > R$.

The number R referred in the theorem above is called the **radius of convergence** of the power series. The radius of convergence is $R = 0$ in the case (a) and $R = \infty$ in the case (b). The set of all values for which the power series converges is called the **interval of convergence** of the power series. Thus, the theorem tells us that the interval of convergence of a power series centered at c is (a) just the single point c, (b) the interval $(-\infty, \infty)$, or (c) the interval $(c-R, c+R)$. But in the last case, the theorem does not tell us whether the endpoints $x = c-R$ and $x = c+R$ are included in the interval of convergence. To determine whether they are included, we simply replace x in the power series by $c-R$ and $c+R$ in succession and use a convergence test on the resultant series.

Example : Find the radius of convergence and the interval of convergence of $\sum_{n=0}^{\infty} n!x^n$.

Solution : We can think of the given as $\sum_{n=0}^{\infty} u_n$, where $u_n = n!x^n$. Applying the Ratio test, we have

$$\lim_{x\to\infty} |\frac{u_{n+1}}{u_n}| = \lim_{n\to\infty} |\frac{(n+1)!x^{n+1}}{n!x^n}| = \lim_{x\to\infty} (n+1)|x| = \infty$$

whenever $x \neq 0$, and we conclude that the series diverges whenever $x \neq 0$. Therefore, the series converges only when $x = 0$, and its radius of convergence is accordingly $R = 0$.

Example : Find the radius of convergence and the interval of convergence of

$$\sim_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Solution : Applying ratio test

$$\lim_{nto\infty} |\frac{(-1)^{n+1}x^{2n+2}}{(2n+2)!} . \frac{(2n)!}{(-1)^n x^{2n}}| = \lim_{n\to\infty} \frac{x^2}{(2n+1)(2n+2)} = 0 < 1$$

for each fixed value of x, so by the Ratio Test, the given series converges for all values of x. Therefore, the radius of convergence of the series is $R = \infty$, and its interval of convergence is $(-\infty, \infty)$.

Example : Find the radius of convergence and the interval of convergence of

$$\lim_{n=1}^{\infty} \frac{x^n}{n}.$$

Solution : Let $u_n = x^n/n$. Then

$$\lim_{n\to\infty} |\frac{u_{n+1}}{u_n}| = \lim_{n\to\infty} |\frac{x^{n+1}}{n+1} . \frac{n}{x^n}| = \lim_{n\to\infty} (\frac{n}{n+1})|x| = |x|$$

By the Ratio Test, the series converges if $-1 < x < 1$. Therefore, the radius of convergence of the series is $R = 1$. To determine the interval of convergence of the power series, we need to examine the behavior of the series at the end-points $x = -1$ and $x = 1$. Now, if $x = -1$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$$

which is the convergent alternating harmonic series, and we see that $x = -1$ is in the interval of convergence of the power series. If $x = 1$, we obtain the harmonic series $\sum_{n=0}^{\infty} 1/n$ which is divergent, so $x = 1$ is not in the interval of convergence. We conclude that the interval of convergence of the given power series is $[-1, 1)$.

Theorem 36. ***(Differentiation and Integration of power series)*** *Suppose that the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ has a radius of convergence $R > 0$. Then the function f defined by*

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + ... + a_n(x-c)^n + ...$$

for all x in $(c-R, c+R)$ is both differentiable and integrable on $(c-R, c+R)$. Moreover, the derivative of f and the indefinite integral of f are

a. $f'(x) = a_1 = 2a_2(x-c) + 3a_3(x-c)^2 + ... = \sum_{n=1}^{\infty} na_n(x-c)^{n-1}$
b. $\int f(x)dx = C + a_0(x-c) + a_1\frac{(x-c)^2}{2} + a_2\frac{(x-c)^3}{3} + ... = \sum_{n=0}^{\infty} \frac{(x-c)^{n+1}}{n+1} + C.$

Notes
1. The series in parts (a) and (b) of above theorem have the same radius of convergence, R, as the series $\sum_{n=0}^{\infty} a_n(x-c)^n$. But the interval of convergence may change. More specifically, you may lose convergence at the endpoints when you differentiate and gain convergence there when you integrate.
2. Above theorem implies that a function that is represented by a power series in an interval $(c-R, c+R)$ is continuous on that interval.

Example : Find a power series representation for $ln(1-x)$ on $(-1,1)$.
Solution : We start with the equation

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... = \sum_{n=0}^{\infty} x^n \quad |x| < 1$$

Integrating both sides of this equation with respect to x, we obtain

$$\int \frac{1}{1-x}dx = \int (1 + x + x^2 + ...)dx$$

$$-ln(1-x) = x + \frac{1}{2}x^2 = \frac{1}{3}x^3 + ... + C$$

To determine the value of C, we set $x = 0$ in this equation to obtain $-ln1 = 0 = C$. Using this value of C, we see that

$$ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - .... = \sum_{n=1}^{\infty} \frac{x^n}{n} \quad |x| < 1$$

Example : Find a power series representation for $\tan^{-1}(x)$ by integrating a power series representation of $f(x) = 1/(1-x^2$.
Solution : Observe that we can obtain a power series representation of f by replacing x with $-x^2$ in the equation

$$\frac{1}{1-x} = 1 + x + x^2 + ... \quad |x| < 1$$

Thus,

$$\begin{aligned}\frac{1}{1+x^2} &= \frac{1}{1-(-x^2)} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 ......\\ &= 1 - x^2 + x^3 - x^6 + ...\\ &= \sum_{n=0}^{\infty}(-1)^n x^{2n}\end{aligned}$$

Since the geometric series converges for $|x| < 1$, we see that this series converges for $|-x^2| < 1$, that is, $x^2 < 1$ or $|x| < 1$. Finally, integrating this equation, we have

$$\begin{aligned}\tan^{-1}(x) &= \int \frac{1}{1+x^2} dx = \int (1 - x^2 + x^4 - x^6 + ...) dx\\ &= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...\end{aligned}$$

To find C, we use the condition $\tan^{-1}(0) = 0$ to obtain $0 = C$.

Therefore,

$$\tan^{-1}(x) = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

3.2 Taylor and Maclaurin series

In the previous section we saw that every power series represents a function whose domain is precisely the interval of convergence of the series. We now look at the general problem of finding power series representations for functions, specifically on what form does the power series representation of the function f take? (In other words, what does a_n look like?)

Theorem 37. ***(Taylor series of*** f ***at*** c***)*** *If* f *has a power series representation at* c*, that is, if*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n \quad |x-c| < R$$

then $f^{(n)}(c)$ *exits for every positive integer* n *and*

$$a_n = \frac{f^{(n)}(c)}{n!}$$

Thus,

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \\ &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + ... \end{aligned}$$

A series of this form is called the Taylor series of the function f at c. In the special case in which $c = 0$, the Taylor series becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n = f(0) + f'(0)(x) + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x-0)^3 + ...$$

This series is just the Taylor series of f centered at the origin. It is called the Maclaurin series of f.

Note : The above theorem states that if a function f has a power series representation at c, then the (unique) series must be the Taylor series at c. The converse is not necessarily true. Given a function f with derivatives of all orders at c, we can compute the Taylor coefficients of f at c,

$$\frac{f^{(n)}(c)}{n!} \quad n = 0, 1, 2, ...$$

and, therefore, the Taylor series of f at c. But the series that is obtained formally in this fashion need not represent f.

Example : Let $f(x) = e^x$. Find the Maclaurin series of f, and determine its radius of convergence.

Solution : The derivatives of $f(x) = e^x$ are $f'(x) = e^x$, $f''(x) = e^x$, and, in general, $f^{(n)}(x) = e^x$, where $n \geq 1$. So

$$f^{(n)}(0) = 1$$

Therefore, the Maclaurin series of f is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

To determine the radius of convergence of the power series, we use the ratio test with $u_n = x^n/n!$. Since

$$\lim_{n\to\infty} |\frac{u_{n+1}}{u_n}| = \lim_{n\to\infty} |\frac{x^{n+1}}{(n+1)!} . \frac{n!}{x^n}| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0$$

we conclude that the radius of convergence of the series is $R = \infty$.

Example : Find the Maclaurin series of $f(x) = sinx$, and determine its interval of convergence.

Solution : To find the Maclaurin series of $f(x) = sinx$, we compute the values of f and its derivatives at $x = 0$. We obtain

$f(x) = sinx \quad f(0) = 0$

$f'(x) = cosx \quad f'(0) = 1$

$f''(x) = -sinx \quad f''(0) = 0$

$f'''(x) = -cosx \quad f'''(0) = -1$

$f^{(4)}(x) = sinx \quad f^{(4)}(0) = 0$

We need not go further, since it is clear that successive derivatives of f follow this same pattern. Then, we obtain the Maclaurin series of $f(x) = sinx$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ...$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}.....$$
$$= \sum_{n=0} \infty \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

To find the interval of convergence of the series, we use the Ratio Test

$$\lim_{n\to\infty} |\frac{u_{n+1}}{u_n}| = \lim_{n\to\infty} |\frac{(-1)^{n+1}x^{2n+3}}{(2n+3)!}.\frac{(2n+1)!}{(-1)^n x^{2n+1}}|$$
$$= \lim_{n\to\infty} \frac{|x|^2}{(2n+2)(2n+3)} = 0$$

we conclude that the interval of convergence of the series is $(-\infty, \infty)$.

Example : Find the Maclaurin series for $f(x) = (1+x)^k$, where k is a real number.

Solution : We compute the values of f and its derivatives at $x = 0$, obtaining

$f(x) = (1+x)^k \quad f(0) = 1$

$f'(x) = k(1+x)^{k-1} \quad f'(0) = k$

$f''(x) = k(k-1)(1+x)^{k-2} \quad f''(0) = k(k-1)$

$f'''(x) = k(k-1)(k-1)(1+x)^{k-3} \quad f'''(0) = k(k-1)(k-2)$

Thus, we get

$f^{(n)}(x) = k(k-1)....(k-n+1)(1+x)^{k-n} \quad f^{(n)}(0) = k(k-1)....(k-n+1)$

So the Maclaurin series of $f(x) = (1+x)^k$ is

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ... \\ &= 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + ... \\ &= \sum_{n=0}^{\infty} \frac{k(k-1)....(k-n+1)}{n!}x^n\end{aligned}$$

Observe that if k is a positive integer, then the series is infinite (by the Binomial Theorem), and so it converges for all x. If k is not a positive integer, then we use the Ratio Test to find the interval of convergence.

$$\begin{aligned}\lim_{n\to\infty} |\frac{u_{n+1}}{u_n}| &= \lim_{n\to\infty} |\frac{k(k-1)(k-2)....(k-n+1)(k-n)x^{n+1}}{(n+1)!} \times \\ &\frac{n!}{k(k-1)(k-2)....(k-n+1)}| \\ &= \lim_{n\to\infty} \frac{|k-n|}{n+1}|x| \\ &= \lim_{n\to\infty} \frac{|\frac{k}{n}-1|}{n+\frac{1}{n}}|x| \\ &= |x|\end{aligned}$$

and we see that the series converges for x in the interval $(-1, 1)$.

The series in example above is called the binomial series.

Definition 27. ***(Binomial series)*** *If k is any real number and*

$|x| < 1$, *then*

$$(1+x)^k = 1+kx+\frac{k(k-1)}{2!}x^2+\frac{k(k-1)(k-2)}{3!}x^3+.... = \sum_{n=0}^{\infty}\binom{k}{n}x^n$$

Notes :

1. The coefficients in the binomial series are referred to as binomial coefficients and are denoted by

$$\binom{k}{n} = \frac{k(k-1)...(k-n+1)}{n!} \quad n \geq 1, \binom{k}{0} = 1$$

2. If k is a positive integer and $n > k$, then the binomial coefficient contains a factor $(k-k)$, so $\binom{k}{n} = 0$ for $n > k$. The binomial series reduces to a polynomial of degree k:

$$(1+x)^k = \sum_{n=0}^{k}\binom{k}{n}x^n$$

In other words, the expression $(1+x)^k$ can be represented by a finite sum if k is a positive integer and by an infinite series if k is not a positive integer. Thus, we can view the binomial series as an extension of the Binomial Theorem to the case in which k is not a positive integer.

3. Even though the binomial series always converges for $-1 < x < 1$, its convergence at the endpoints $x = -1$ or $x = 1$ depends on the value of k. It can be shown that the series converges at $x = 1$ if $-1 < k < 0$ and at both endpoints $x = \pm 1$ if $k \geq 0$.

Techniques for Finding Taylor Series We have already seen how to find Taylor series of any function using the equation described above. But it is often easier to find the series by algebraic manipulation, differentiation, or integration of some well-known series. We now elaborate further on such techniques. First, we list some common functions and their power series representations.

Maclaurin Series	Interval of Convergence
1. $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$	$(-1, 1)$
2. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	$(-\infty, \infty)$
3. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$(-\infty, \infty)$
4. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$	$(-\infty, \infty)$
5. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$	$(-1, 1]$
6. $\sin^{-1} x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \cdots = \sum_{n=0}^{\infty} \frac{(2n)!\, x^{2n+1}}{(2^n n!)^2(2n+1)}$	$[-1, 1]$
7. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	$[-1, 1]$
8. $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$	$(-1, 1)$

Example : Find the Taylor series representation of $f(x) = \frac{1}{1+x}$ at $x = 2$.

Solution : We first rewrite $f(x)$ so that it includes the expression $(x-2)$. Thus,

$$f(x) = \frac{1}{1+x} = \frac{1}{3+(x-2)} = \frac{1}{3[1+(\frac{x-2}{3})]} = \frac{1}{3}.\frac{1}{1+(\frac{x-2}{3})}$$

Then, using Formula in table above with x replaced by $-(x -$

2)/3, we obtain

$$\begin{aligned}
f(x) &= \frac{1}{3}[\frac{1}{1-(-(\frac{x-2}{3}))}] \\
&= \frac{1}{3}[1+(-(\frac{x-2}{3}))+(-(\frac{x-2}{3}))^2+(-(\frac{x-2}{3}))^3+.... \\
&= \frac{1}{3}[1-(\frac{x-2}{3})+(\frac{x-2}{3})^2-(\frac{x-2}{3})^3+....] \\
&= \frac{1}{3} = \frac{1}{3^2}(x-2)+\frac{1}{3^3}(x-2)^2-\frac{1}{3^4}(x-2)^3+.... \\
&= \sum_{n=0}^{\infty}(-1)^b\frac{(x-2)^n}{3^{n+1}}
\end{aligned}$$

The series converges for $|(x-2)/3| < 1$, that is, $|x-2| < 3$ or $-1 < x < 5$.

Example : Find the Taylor series for $f(x) = sinx$ at $x = \pi/6$.
Solution : We write

$$\begin{aligned}
f(x) = sinx &= sin[(x-\frac{\pi}{6})+\frac{\pi}{6}] \\
&= sin(x-\frac{\pi}{6})cos(\frac{\pi}{6})+cos(x-\frac{\pi}{6})sin(\frac{\pi}{6}) \\
&= \frac{\sqrt{3}}{2}sin(x-\frac{\pi}{6})+\frac{1}{2}cos(x-\frac{\pi}{6})
\end{aligned}$$

Then using Formulas 3 and 4 with $x-(\pi/6)$ in place of x, we obtain

$$f(x) = \frac{\sqrt{3}}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(x-\frac{\pi}{6})^{2n+1}+\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(x-\frac{\pi}{6})^{2n}$$

which converges for all x in $(-\infty,\infty)$.

The power series representations of certain functions can also be found by adding, multiplying, or dividing the Maclaurin or Taylor series of some familiar functions as the following examples show.

Example : Find the Maclaurin series representation for $f(x) = \sinh(x)$.

Solution : We have

$$\begin{aligned}
f(x) &= \frac{1}{2(e^x - e^{-x})} = \frac{1}{2}e^x - \frac{1}{2}e^{-x} \\
&= \frac{1}{2}(1 + x + \frac{x^2}{2!}) + \frac{x^3}{3!} + ...) - \frac{1}{2}(1 - x + \frac{x^2}{2!}) - \frac{x^3}{3!} + ...) \\
&= x + \frac{x^3}{3!} + \frac{x^5}{5!} + ... \\
&= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}
\end{aligned}$$

Since the Maclaurin series of both e^x and e^{-x} converge for x in $(-\infty, \infty)$, we see that this representation of $\sinh(x)$ is also valid for all values of x.

We can also use Taylor series to integrate functions whose anti derivatives cannot be found in terms of elementary functions. Examples of such functions are e^{-x^2} and $sinx^2$. In particular, the use of Taylor series enables us to obtain approximations to definite integrals involving such functions, as illustrated in the following example.

Example : Find $\int e^{-x^2} dx$.

Example : Replacing x in Formula (2) in Table by $-x^2$ gives

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + ... = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

Integrating both sides of this equation with respect to x, we obtain,

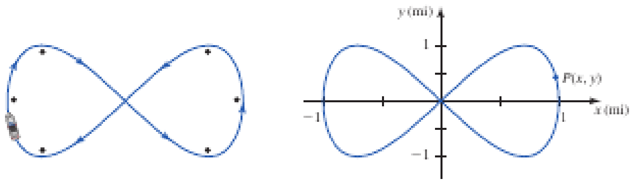
$$\begin{aligned} \int e^{-x^2} dx &= \int (1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + ...) dx \\ &= C + x - \frac{1}{3}x^3 + \frac{1}{5.2!}x^5 - \frac{1}{7.3!}x^7 + ... \\ &= C + \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1).n!} x^{2n+1} \end{aligned}$$

Since the power series representation of e^{-x^2} converges for x in $(-\infty, \infty)$, this result is valid for all values of x.

3.3 Plane curves and Parametric equations

Why we use parametric equations :

Figure below gives a bird's-eye view of a proposed training course for a yacht. In Figure we have introduced an xy-coordinate system in the plane to describe the position of the yacht. With respect to this coordinate system the position of the yacht is given by the point $P(x, y)$, and the course itself is the graph of the rectangular equation $4x^4 - 4x^2 + y^2 = 0$, which is called a lemniscate. But representing the lemniscate in terms of a rectangular equation in this instance has three major drawbacks.



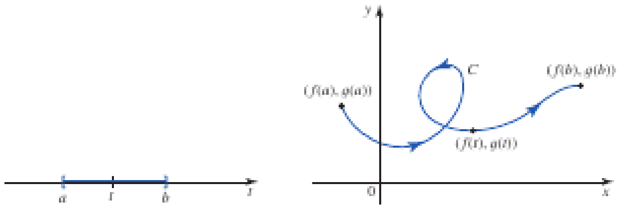
(a) The dots give the position of markers. (b) An equation of the curve C is $4x^4 - 4x^2 + y^2 = 0$.

First, the equation does not define y explicitly as a function of x or x as a function of y. You can also convince yourself that this is not the graph of a function by applying the vertical and horizontal line tests to the curve in the figure. Because of this, we cannot make direct use of many of the results for functions. Second, the equation does not tell us when the yacht is at a given point (x, y). Third, the equation gives no inkling as to the direction of motion of the yacht. To overcome these drawbacks when we consider the motion of an object in the plane or plane curves that are not graphs of functions, we turn to the following representation. If (x, y) is a point on a curve in the xy-plane, we write

$$x = f(t) \quad y = g(t)$$

where f and t are functions of an auxiliary variable t with (common) domain some interval I. These equations are called **parametric equations**, t is called a **parameter**, and the interval I is called a **parameter interval**.

If we think of t on the closed interval $[a, b]$ as representing time, then we can interpret the parametric equations in terms of the motion of a particle as follows: At $t = a$ the particle is at the **initial point** $(f(a), g(a))$ of the curve or **trajectory** C. As t increases from $t = a$ to $t = b$, the particle traverses the curve in a specific direction called the **orientation** of the curve, eventually ending up at the **terminal point** $(f(b), g(b))$ of the curve. (See Figure below.)



Parameter interval is $[a, b]$.

Sketching curves defined by parametric equations

Definition 28. ***(Plane curve)*** *A plane curve is a set C of ordered pairs (x, y) defined by the parametric equations*

$$x = f(t) \quad and \quad y = g(t)$$

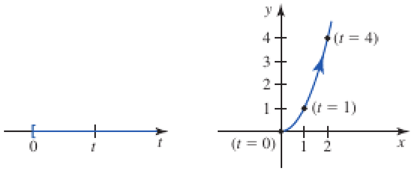
where f and g are continuous functions on a parameter interval I.

Example : Sketch the curves represented by

a. $x = \sqrt{t}$ and $y = t$

b. $x = t$ and $y = t^2$

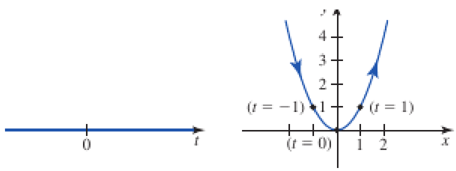
Solution : a. We eliminate the parameter t by squaring the first equation to obtain $x^2 = t$. Substituting this value of t into the second equation, we obtain $y = x^2$, which is an equation of a parabola. But note that the first parametric equation implies that $t \geq 0$, so $x \geq 0$. Therefore, the desired curve is the right portion of the parabola shown in Figure below. Finally, note that the parameter interval is $[0, \infty)$, and as t increases from 0, the desired curve starts at the initial point $(0, 0)$ and moves away from it along the parabola.



Parameter interval is $[0, \infty)$.

t	(x, y)
0	(0, 0)
1	(1, 1)
2	$(\sqrt{2}, 2)$
4	(2, 4)

b. Substituting the first equation into the second yields $y = x^2$. Although the rectangular equation is the same as that in part (a), the curve described by the parametric equations here is different from that of part (a), as we will now see. In this instance the parameter interval is $(-\infty, \infty)$. Furthermore, as t increases from $-\infty$ to ∞ , the curve runs along the parabola $y = x^2$ from left to right, as you can see by plotting the points corresponding to, say, $t = -1, 0$, and 1. You can also see this by examining the parametric equation $x = t$, which tells us that x increases as t increases. (See Figure below.)



Parameter interval is $(-\infty, \infty)$.

t	(x, y)
−1	(−1, 1)
0	(0, 0)
1	(1, 1)

Example : Describe the curve represented by

$$x = 4cos\theta \quad and \quad y = 3sin\theta \quad 0 \leq \theta \leq 2\pi$$

Solution : Solving the first equation for $cos\theta$ and the second equation for $sin\theta$ gives

$$cos\theta = \frac{x}{4}$$

and

$$sin\theta = \frac{y}{3}$$

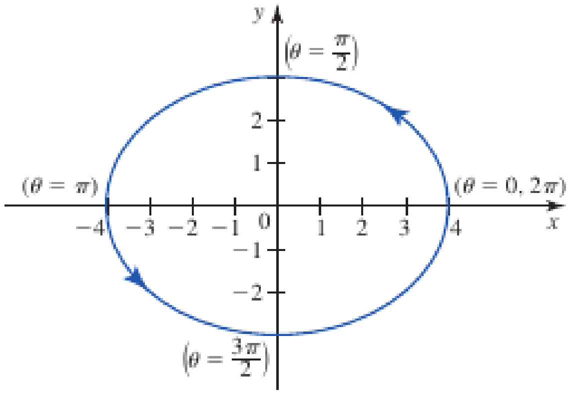
Squaring each equation and adding the resulting equations, we obtain

$$cos^2\theta + sin^2\theta = (\frac{x}{4})^2 + (\frac{y}{3})^2$$

Since $cos^2\theta + sin^2\theta = 1$, we end up with the rectangular equations

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

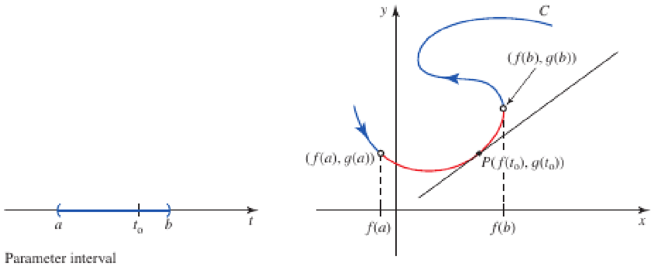
From this we see that the curve is contained in an ellipse centered at the origin. If $\theta = 0$, then $x = 4$ and $y = 0$, giving $(4, 0)$ as the initial point of the curve. As θ increases from 0 to 2π, the elliptical curve is traced out in a counterclockwise direction, terminating at $(4, 0)$. (See Figure below.)



3.4 The calculus of Parametric equations

Tangent line to curves defined by parametric equation

Suppose that C is a smooth curve that is parametrized by the equations $x = f(t)$ and $y = g(t)$ with parameter interval I and we wish to find the slope of the tangent line to the curve at the point P. (See Figure below.) Let t_0 be the point in I that corresponds to P, and let (a, b) be the subinterval of I containing t_0 corresponding to the highlighted portion of the curve C in the figure. This subset of C is the graph of a function of x, as you can verify using the Vertical Line Test.



Let's denote this function by F so that $y = F(x)$, where $f(a) < x < f(b)$. Since $x = f(t)$ and $y = g(t)$, we may rewrite this equation in the form

$$g(t) = F[f(t)]$$

Using the Chain Rule, we obtain

$$\begin{aligned} g'(t) &= F'[f(t)]f'(t) \\ &= F'(x)f'(t) \end{aligned}$$

If $f'(t) \neq 0$, we can solve for $F'(x)$, obtaining

$$F'(x) = \frac{g'(t)}{f'(x)}$$

which we can write as

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad if \ \frac{dx}{dt} \neq 0$$

Horizontal and vertical Tangents

A curve C represented by the parametric equations $x = f(t)$ and $y = g(t)$ has a horizontal tangent at a point (x, y) on C where $dy/dt = 0$ and $dx/dt \neq 0$ and a vertical tangent where $dx/dt = 0$ and $dy/dt \neq 0$, so that dy/dx is undefined there. Points where both dy/dt and dx/dt are equal to zero are candidates for horizontal or vertical tangents and may be investigated by using l'Hopital's rule.

Example : A curve C is defined by the parametric equations $x = t^2$ and $y = t^3 - 3t$

a. Find the points on C where the tangent lines are horizontal or vertical.

b. Find the x- and y-intercepts of C.

Solution :a. Setting $dy/dt = 0$ gives $3t^2 - 3 = 0$, or $t = \pm 1$. Since $dx/dt = 2t \neq 0$ at these values of t, we conclude that C has horizontal tangents at the points on C corresponding to $t = \pm 1$, that is, at $(1, -2)$ and $(1, 2)$. Next, setting $dx/dt = 0$ gives $2t = 0$, or $t = 0$. Since $dy/dt \neq 0$ for this value of t, we conclude that C has a vertical tangent at the point corresponding to $t = 0$, or at $(0, 0)$.

b. To find the x-intercepts, we set $y = 0$, which gives $t^3 - 3t = t(t^2 - 3) = 0$, or $t = -\sqrt{3}, 0$, and $\sqrt{3}$. Substituting these values of t into the expression for x gives 0 and 3 as the x-intercepts. Next, setting $x = 0$ gives $t = 0$, which, when substituted into the expression for y, gives 0 as the y-intercept.

Finding $\frac{d^2y}{dx^2}$ from parametric equations

Suppose that the parametric equations $x = f(t)$ and $y = g(t)$ define y as a twice-differentiable function of x over some suitable interval. Then d^2y/dx^2 may be found from Equation of dy/dx with another application of the Chain Rule.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{dy}{dx}) = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}} \quad if \frac{dx}{dt} \neq 0$$

Higher-order derivatives are found in a similar manner.

The length of a smooth curve

Theorem 38. ***(Length of a smooth curve)*** *Let C be a smooth curve represented by the parametric equations $x = f(t)$ and $y = g(t)$ with parameter interval $[a, b]$. If C does not intersect itself, except possibly for t $= a$ and t $= b$, then the length of C is*

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt$$

The area of a surface of revolution

Theorem 39. ***(Area of surface of revolution)*** *Let C be a smooth curve represented by the parametric equations $x = f(t)$ and $y = g(t)$ with parameter interval $[a, b]$, and suppose that C does not intersect itself, except possibly for t $= a$ and t $= b$. If $g(t) \geq 0$ for all t in $[a, b]$, then the area S of the surface obtained by revolving C about the x-axis is*

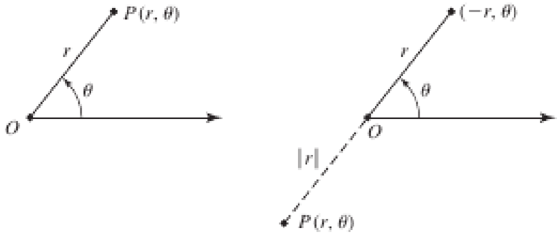
$$S = 2\pi \int_a^b y\sqrt{[f'(t)]^2 + [g'(t)]^2} = 2\pi \int_a^b y\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt$$

If $f(t) \geq 0$ for all t in $[a, b]$, then the area S of the surface that is obtained by revolving C about the y-axis is

$$S = 2\pi \int_a^b x\sqrt{[f'(t)]^2 + [g'(t)]^2} = 2\pi \int_a^b x\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt$$

3.5 Polar coordinates

The polar coordinate system : To construct the polar coordinate system, we fix a point O called the **pole** (or **origin**) and draw a ray (half-line) emanating from O called the polar axis. Suppose that P is any point in the plane, let r denote the distance from O to P, and let θ denote the angle (in degrees or radians) between the polar axis and the line segment OP. Then the point P is represented by the ordered pair (r, θ), also written $P(r, \theta)$, where the numbers r and θ are called the **polar coordinates** of P. The **angular coordinate** θ is positive if it is measured in the counterclockwise direction from the polar axis and negative if it is measured in the clockwise direction. The **radial coordinate** r may assume positive as well as negative values. If $r > 0$, then $P(r, \theta)$ is on the terminal side of θ and at a distance r from the origin. If $r < 0$, then $P(r, \theta)$ lies on the ray that is opposite the terminal side of θ and at a distance of $|r| = -r$ from the pole. Also, by convention the pole O is represented by the ordered pair $(0, \theta)$ for any value of θ. Finally, a plane that is endowed with a polar coordinate system is referred to as an $r\theta$-plane.



(a) $r > 0$ (b) $r < 0$

Note : Unlike the representation of points in the rectangular system, the representation of points using polar coordinates is not unique. For example, the point (r, θ) can also be written as $(r, \theta + 2n\pi)$ or $(-r, \theta + (2n + 1)\pi)$, where n is any integer.

Relationship between Polar and Rectangular coordinates

Suppose that a point P (other than the origin) has representation (r, θ) in polar coordinates and (x, y) in rectangular coordinates. Then

$$x = rcos\theta \quad and \quad y = rsin\theta$$

$$r^2 = x^2 + y^2 \quad and \quad than\theta = \frac{y}{x} \;\; if \;\; x \neq 0$$

Example : The point $(-1, 1)$ is given in rectangular coordinates. Find its representation in polar coordinates.

Solution : Here, $x = -1$ and $y = 1$, Then, we have

$$r^2 = x^2 + y^2 = (-1)^2 + 1^2 = 2$$

and

$$tan\theta = \frac{y}{x} = -1$$

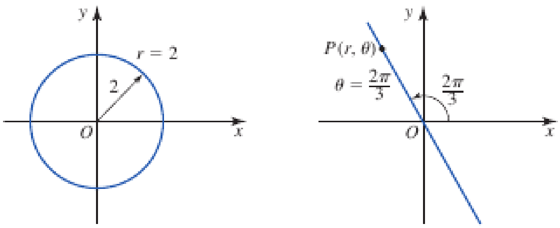
Let's choose r to be positive; that is, $r = \sqrt{2}$. Next, observe that the point $(-1, 1)$ lies in the second quadrant and so we choose $\theta = 3\pi/4$ (other choices are $\theta = (3\pi/4) \pm 2n\pi$, where n is an integer). Therefore, one representation of the given point is $(\sqrt{2}, \frac{3\pi}{4})$.

Graphs of polar equations

The graph of a polar equation $r = f(\theta)$ or, more generally, $F(r, \theta) = 0$ is the set of all points (r, θ) whose coordinates satisfy the equation.

Example : Sketch the graphs of the polar equations, and reconcile your results by finding the corresponding rectangular equations.
a. $r = 2$
b. $\theta = \frac{2\pi}{3}$
Solution : a. The graph of $r = 2$ consists of all points $P(r, \theta)$ where $r = 2$ and θ can assume any value. Since r gives the distance between P and the pole O, we see that the graph consists of all points that are located a distance of 2 units from the pole; in other words, the graph of $r = 2$ is the circle of radius 2 centered at the pole.(See Figure (a) below.) To find the corresponding rectangular equation, square both sides of the given equation obtaining $r^2 = 4$. Then, we have $r^2 = x^2 + y^2$, and this gives the desired equation $x^2 + y^2 = 4$. Since this is a rectangular equation of a

circle with center at the origin and radius 2, the result obtained earlier has been confirmed.



(a) The graph of $r = 2$ (b) The graph of $\theta = \frac{2\pi}{3}$

b. The graph of $\theta = 2\pi/3$ consists of all points $P(r, \theta)$ where $u = 2\pi/3$ and r can assume any value. Since θ measures the angle the line segment OP makes with the polar axis, we see that the graph consists of all points that are located on the straight line passing through the pole O and making an angle of $2\pi/3$ radians with the polar axis.(See Figure (b) above.) Observe that the half-line in the second quadrant consists of points for which $r > 0$, whereas the half-line in the fourth quadrant consists of points for which $r < 0$. To find the corresponding rectangular equation, we use the equation, $tan\theta = y/x$, to obtain

$$tan\frac{2\pi}{3} = \frac{y}{x} \quad or \quad \frac{y}{x} = -\sqrt{3}$$

or $y = -\sqrt{3}x$. This equation confirms that the graph of $\theta = 2\pi/3$ is a straight line with slope $-\sqrt{3}$.

Symmetry

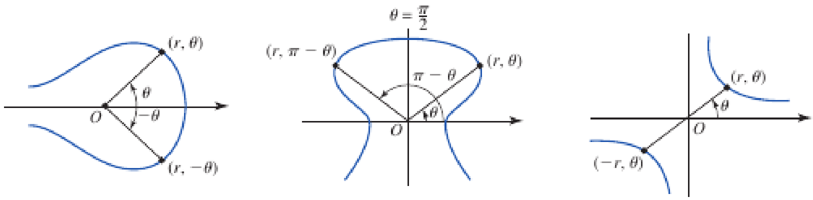
Test for Symmetry

a. The graph of $r = f(\theta)$ is symmetric with respect to the polar axis if the equation is unchanged when θ is replaced by $-\theta$.

b. The graph of $r = f(\theta)$ is symmetric with respect to the vertical line

$theta = \pi/2$ if the equation is unchanged when θ is replaced by $\pi - \theta$.

c. The graph of $r = f(\theta)$ is symmetric with respect to the pole if the equation is unchanged when r is replaced by $-r$ or when θ is replaced by $\theta - \pi$.



(a) Symmetry with respect to the polar axis

(b) Symmetry with respect to the line $\theta = \frac{\pi}{2}$

(c) Symmetry with respect to the pole

Tangent lines to graph of polar equations

To find the slope of the tangent line to the graph of $r = f(\theta)$ at the point $P(r, \theta)$, let $P(x, y)$ be the rectangular representation of P. Then

$$x = rcos\theta = f(\theta)cos\theta$$

$$y = rsin\theta = f(\theta)sin\theta$$

We can view these equations as parametric equations for the graph of $r = f(\theta)$ with parameter θ. Then, we have

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dy}{d\theta} sin\theta + rcos\theta}{\frac{dr}{d\theta} cos\theta - rsin\theta} \quad if \frac{dx}{d\theta} \neq 0$$

and this gives the slope of the tangent line to the graph of $r = f(\theta)$ at any point $P(r, \theta)$.

The horizontal tangent lines to the graph of $r = f(\theta)$ are located at the points where $dy/d\theta = 0$ and $dx/d\theta \neq 0$. The vertical tangent lines are located at the points where $dx/d\theta = 0$ and $dy/d\theta \neq 0$ (so that dy/dx is undefined). Also, points where both $dy/d\theta$ and $dx/d\theta$ are equal to zero are candidates for horizontal or vertical tangent lines, respectively, and may be investigated using l'Hopital's Rule.

Example : Find the tangent lines of $r = cos2\theta$ at the origin.
Solution : Setting $f(\theta) = cos2\theta = 0$, we find that

$$2\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, or\ \frac{7\pi}{2}$$

or

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, or\ \frac{7\pi}{4}$$

Next, we compute $f'(\theta) = -2sin2\theta$. Since $f'(\theta) \neq 0$ for each of these values of θ, we see that $\theta = \pi/4$ and $\theta = 3\pi/4$ (that is, $y = x$ and $y = -x$) are tangent lines to the graph of $r = cos2\theta$ at the pole.

3.6 Areas and Arc lengths in Polar coordinates

Areas in Polar coordinates

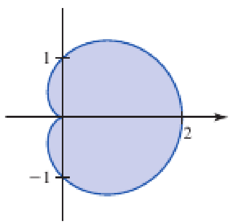
Theorem 40. *(**Area bounded by a Polar curve**) Let f be a continuous, nonnegative function on $[\alpha, \beta]$ where $0 \leq \beta - \alpha < 2\pi$. Then the area A of the region bounded by the graphs of $r = f(\theta)$, $\theta = \alpha$, and $\theta = \beta$ is given by*

$$A = \int_{\alpha}^{\beta} \frac{1}{2}[f(\theta)]^2 d\theta = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$$

Note : When you determine the limits of integration, keep in mind that the region R is swept out in a counterclockwise direction by the ray emanating from the origin, starting at the angle α and terminating at the angle β.

Example : Find the area of the region enclosed by the cardioid $r = 1 + cos\theta$.

Solution :



The graph of the cardioid $r = 1 + cos\theta$ is shown in the figure above. Observe that the ray emanating from the origin sweeps out the required region exactly once as θ increases from 0 to 2π. Therefore, the required area A is

$$\begin{aligned}
A &= \int_0^{2\pi} \frac{1}{2} r^2 d\theta = \int_0^{2\pi} \frac{1}{2}(1+cos\theta)^2 d\theta \\
&= \frac{1}{2}\int_0^{2\pi} (1+2cos\theta+cos^2\theta)d\theta \\
&= \frac{1}{2}\int_0^{2\pi} (1+2cos\theta+\frac{1+cos2\theta}{2})d\theta \\
&= \frac{1}{2}\int_0^{2\pi} (\frac{3}{2}+2cos\theta+\frac{1}{2}cos2\theta)d\theta \\
&= \frac{1}{2}[\frac{3}{2}+2sin\theta+\frac{1}{4}sin2\theta]_0^{2\pi} = \frac{3}{2}\pi
\end{aligned}$$

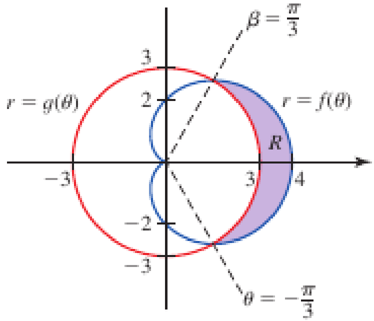
Area bounded by two graphs

Theorem 41. ***(Area bounded by two polar curves)*** *Let f and g be continuous on $[\alpha, \beta]$, where $0 \leq g(\theta) \leq f(\theta)$ and $0 \leq \beta - \alpha < 2\pi$. Then the area A of the region bounded by the graphs of $r = g(\theta)$, $r = f(\theta)$, $\theta = \alpha$, and $\theta = \beta$ is given by*

$$A = \frac{1}{2}\int_\alpha^\beta ([f(\theta)]^2 - [g(\theta)]^2)d\theta$$

Example : Find the area of the region that lies outside the circle $r = 3$ and inside the cardioid $r = 2 + 2cos\theta$.

Solution :



We first sketch the circle $r = 3$ and the cardioid $r = 2 + 2cos\theta$. The required region is shown shaded in the figure above. To find the points of intersection of the two curves, we solve the two equations simultaneously. We have $2 + 2cos\theta = 3$ or $cos\theta = 1/2$, which gives $\theta = \pm\pi/3$. Since the region of interest is swept out by the ray emanating from the origin as θ varies from $-\pi/3$ to $\pi/3$, we see that the required area is,

$$A = \frac{1}{2}\int_{\alpha}^{\beta}([f(\theta)]^2 - [g(\theta)]^2)d\theta$$

where $f(\theta) = 2 + 2cos\theta = 2(1 + cos\theta)$, $g(\theta) = 3$, $\alpha = -\pi/3$, and $\beta = \pi/3$. If we take advantage of symmetry, we can write

$$\begin{aligned}
A &= 2(\frac{1}{2}\int_0^{\pi/3}([2(1+cos\theta)]^2 - 3^2)d\theta \\
&= \int_0^{\pi/3}(4 + 8cos\theta + 4cos^2\theta - 9)d\theta \\
&= \int_0^{\pi/3}(-5 + 8cos\theta + 4.\frac{1+cos2\theta}{2})d\theta \\
&= \int_0^{\pi/3}(-3 + 8cos\theta + 2cos2\theta)d\theta \\
&= [-3\theta + 8sin\theta + sin2\theta]_0^{\pi/3} \\
&= (-\pi + 8(\frac{\sqrt{3}}{2}) + \frac{\sqrt{3}}{2}) \\
&= \frac{9\sqrt{3}}{2} - \pi
\end{aligned}$$

Arc length in polar coordinates

Theorem 42. ***(Arc length)*** *Let f be a function with a continuous derivative on an interval $[\alpha, \beta]$. If the graph C of $r = f(\theta)$ is traced exactly once as θ increases from α to β, then the length L of C is given by*

$$L = \int_\alpha^\beta \sqrt{[f'(\theta)]^2 + [f(\theta)]^2}d\theta = \int_\alpha^\beta \sqrt{(\frac{dr}{d\theta})^2 + r^2}d\theta$$

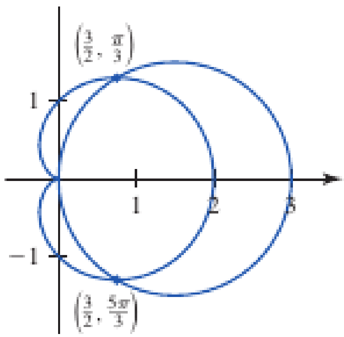
Area of surface of revolution

Theorem 43. ***(Area of Surface of revolution)*** *Let f be a function with a continuous derivative on an interval $[\alpha, \beta]$. If the graph C of $r = f(\theta)$ is traced exactly once as θ increases from α to β, then the area of the surface obtained by revolving C about the*

indicated line is given by
$a.S = 2\pi \int_{\alpha}^{\beta} rsin\theta\sqrt{(\frac{dr}{d\theta})^2 + r^2}d\theta$ *(about the polar axis)*
$b.S = 2\pi \int_{\alpha}^{\beta} rcos\theta\sqrt{(\frac{dr}{d\theta})^2 + r^2}d\theta$ *(about the lie $\theta = \pi/2$)*

Points of intersection of graphs in polar coordinates

In a previous example we were able to find the points of intersection of two curves with representations in polar coordinates by solving a system of two equations simultaneously. This is not always the case. Consider for example, the graphs of the cardioid $r = 1 + cos\theta$ and the circle $r = 3cos\theta$ shown in figure below.



Solving the two equations simultaneously, we obtain

$$3cos\theta = 1 + cos\theta$$

$$cos\theta = \frac{1}{2}$$

or $\theta = \pi/3$ and $5\pi/3$. Therefore, the point of intersection are $(3/2, \pi/3)$ and $(3/2, 5\pi/3)$. But one glance at the figure shows the pole as a third point of intersection that is not revealed in our calculation. To see how this can happen, think of the cardioid as being traced by the point (r, θ) satisfying

$$r = f(\theta) = 1 + cos\theta \quad 0 \leq \theta \leq 2\pi$$

with θ as a parameter. If we think of θ as representing time, then as θ runs from $\theta = 0$ through $\theta = 2\pi$, the point (r, θ) starts at $(2, 0)$ and traverses the cardioid in a counter-clockwise direction, eventually returning to the point $(2, 0)$. Similarly, the circle is traced twice in the counterclockwise direction, by the point (r, θ), where

$$r = g(\theta) = 3cos\theta \quad 0 \leq \theta \leq 2\pi$$

and the parameter θ, once again representing time, runs from $\theta = 0$ through $\theta = 2\pi$.

Observe that the point tracing the cardioid arrives at the point $(3/2, \pi/3)$ on the cardioid at precisely the same time the point tracing the circle arrives at the point $(3/2, \pi/3)$ on the circle. A similar observation holds at the point $(3/2, 5\pi/3)$ on each of the two curves. These are the points of intersection found earlier.

Next, observe that the point tracing the cardioid arrives at the origin when $\theta = \pi$. But the point tracing the circle first arrives at the origin when $\theta = \pi/2$ and then again when $u = 3\pi/2$. In other words, these two points arrive at the origin at different times, so there is no (common) value of θ corresponding to the origin that satisfies both equations simultaneously. Thus, although the origin is a point of intersection of the two curves, this fact will not show up in the solution of the system of equations. For this reason it is recommended that we sketch the graphs of polar equations when finding their points of intersection.

Example : Find the points of intersection of $r = cos\theta$ and $r = cos2\theta$.

Solution : We solve the system of equations

$$r = cos\theta$$

$$r = cos2\theta$$

We set $cos\theta = cos2\theta$ and use the identity $cos2\theta = 2cos2\theta - 1$. We obtain

$$2cos^2\theta - cos\theta = 0$$

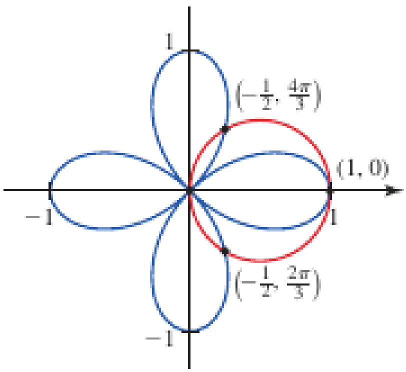
$$(2cos\theta + 1)(cos\theta - 1) = 0$$

So

$$cos\theta = -\frac{1}{2} \quad or \quad cos\theta = 1$$

that is,

$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \ or\ 0$$



These values of θ give $(-1/2, 2\pi/3)$, $(-1/2, 4\pi/3)$, and $(1, 0)$ as the points of interaction. Since both graphs also pass through the pole, we conclude that the pole is also a point of intersection.

Module 4

Geometry of Space and Vector-valued function

4.1 Equations of Lines in Space

Definition 29. ***(Parametric equation of line)*** *The parametric equations of the line passing through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector* **v** $= (a, b, c)$ *are*
$x = x_0 + at$, $y = y_0 + bt$, *and* $z = z_0 + ct$

Example : Find parametric equations for the line passing through the point $P_0(-2, 1, 3)$ and parallel to the vector **v** $= \langle 1, 2, -2 \rangle$.
Solution : Using the above equation with $x_0 = -2$, $y_0 = 1$, $z_0 = 3$, $a = 1$, $b = 2$, and $c = -2$, we obtain
$x = -2 + t$, $y = 1 = 2t$, and $z = 3 - 2t$

Suppose that the vector $v = \langle a, b, c \rangle$ defines the direction of a line L. Then the numbers a, b, and c are called the **direction numbers** of L. Observe that if a line L is described by a set of parametric equations, then the direction numbers of L are precisely the coefficients of t in each of the parametric equations.

There is another way of describing a line in space.

Definition 30. ***(Symmetric equations of a Line)*** *The* ***symmetric equations*** *of the line L passing through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector* **v** $= \langle a, b, c\rangle$ *are*

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Note: Suppose $a = 0$ and both b and c are not equal to zero, then the parametric equations of the line take the form
$x = x_0$, $y = y_0 + bt$, and $z = z_0 + ct$
and the line lies in the plane $x = x_0$ (parallel to the yz-plane). Solving the second and third equations for t leads to

$$x = x_0, \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

which are the symmetric equations of the line.

Example : a. Find parametric equations and symmetric equations for the line L passing through the points $P(-3, 3, -2)$ and $Q(2, -1, 4)$.
b. At what point does L intersect the xy-plane?
Solution : a. The direction of L is the same as that of the vector $\vec{PQ} = \langle 5, -4, 6\rangle$. Since L passes through $P(-3, 3, -2)$, we can use $a = 5$, $b = -4$, $c = 6$, $x_0 = -3$, $y_0 = 3$, and $z_0 = -2$, to obtain the parametric equations

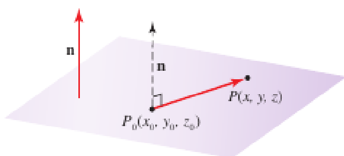
$x = -3 + 5t$, $y = 3 - 4t$, and $z = -2 + 6t$
Thus, we obtain the following symmetric equations for L:

$$\frac{x+3}{5} = \frac{y-3}{-4} = \frac{z+2}{6}$$

b. At the point where the line intersects the xy-plane, we have $z = 0$. So setting $z = 0$ in the third parametric equation, we obtain $t = 1/3$. Substituting this value of t into the other parametric equations gives the required point as $(-4/3, 5/3.0)$.

Equations of Planes in space:

A plane in space is uniquely determined by specifying a point $P_0(x_0, y_0, z_0)$ lying in the plane and a vector **n** $=\langle a, b, c\rangle$ that is normal (perpendicular) to it. (See the figure below)



Definition 31. *(**The standard** form of the equations of a plane) The standard form of the equation of a plane containing the point $P_0(x_0, y_0, z_0)$ and having the normal vector **n** = $\langle a, b, c\rangle$ is*

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

By expanding the equation above and regrouping the terms, we obtain the general form of the equation of a plane in space,

$$ax + by + cz = d$$

where $d = ax_0 + by_0 + cz_0$. Conversely, given $ax + by + cz = d$ with a, b, and c not all equal to zero, we can choose numbers x_0, y_0, and z_0 such that $ax_0 + by_0 + cz_0 = d$.

An equation of the form $ax + by + cz = d$, with a, b, and c not all zero, is called **linear equation in the three variables** x, y, and z.

Theorem 44. *Every plane in space can be represented by a linear equation $ax + by + cz = d$, where a, b, and c are not all equal to zero. Conversely, every linear equation $ax + by + cz = d$ represents a plane in space having a normal vector $\langle a, b, c\rangle$.*

Note: Notice that the coefficients of x, y, and z are precisely the components of the normal vector **n** = $\langle a, b, c\rangle$. Thus, we can write a normal vector to a plane by simply inspecting its equation.

Example : Find an equation of the plane containing the points $P(3, -1, 1)$, $Q(1, 4, 2)$, and $R(0, 1, 4)$.

Solution : We need to find a vector normal to the plane in question. Observe that both of the vectors $\vec{PQ} = \langle -2, 5, 1\rangle$ and $\vec{PR} = \langle -3, 2, 3\rangle$ lie in the plane, so the vector $\vec{PQ} \times \vec{PR}$ is normal to the plane. Denoting this vector by **n**, we have

$$\mathbf{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 5 & 1 \\ -3 & 2 & 3 \end{vmatrix} = 13\mathbf{i} + 3\mathbf{j} + 11\mathbf{k}$$

Finally, using the point $P(3, -1, 1)$ in the plane (any of the other two points will also do) and the normal vector **n** just found, with $a = 13$, $b = 3$, $c = 11$, $x_0 = 3$, $y_0 = -1$, and $z_0 = 1$, gives

$$13(x - 3) + 3(y + 1) + 11(z - 1) = 0$$

or

$$13x + 3y + 11z = 47$$

Parallel and orthogonal planes

Two planes with normal vectors **m** and **n** are parallel to each other if **m** and **n** are parallel; the planes are orthogonal if **m** and **n** are orthogonal.

Example : Find an equation of the plane containing $P(2, -1, 3)$ and parallel to the plane defined by $2x - 3y + 4z = 6$.
Solution : The normal vector of the given plane is **n** = $\langle 2, -3, 4 \rangle$. Since the required plane is parallel to the given plane, it also has **n** as a normal vector. Therefore, we obtain

$$2(x - 2) - 3(y + 1) + 4(z - 3) = 0$$

or

$$2x - 3y + 4z = 19$$

as an equation of the plane.

The angle between two plane

Two distinct planes in space are either parallel to each other or intersect in a straight line. If they do intersect, then the angle between the two planes is defined to be the acute angle between their normal vectors.

Example : Find the angle between the two planes defined by $3x - y + 2z = 1$ and $2x + 3y - z = 4$.

Solution : The normal vectors of these planes are

$\mathbf{n_1} = \langle 3, -1, 2 \rangle$ and $\mathbf{n_2} = \langle 2, 3, -1 \rangle$

Therefore, the angle θ between the planes is given by

$$\begin{aligned} cos\theta &= \frac{\mathbf{n_1}.\mathbf{n_2}}{|\mathbf{n_1}||\mathbf{n_2}|} \\ &= \frac{\langle 3, -1, 2 \rangle . \langle 2, 3, -1 \rangle}{\sqrt{9+1+4}\sqrt{4+9+1}} \\ &= \frac{3(2) + (-1)(3) + 2(-1)}{\sqrt{14}\sqrt{14}} = \frac{1}{14} \end{aligned}$$

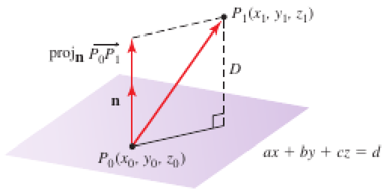
or

$$\theta = cos^{-1}(\frac{1}{14}) \approx 80^\circ$$

The distance between a point and a plane

Suppose that $P_1(x_1, y_1, z_1)$ is a point not lying in the plane $ax + by + cz = d$. Let $P_0(x_0, y_0, z_0)$ be any point lying in the plane. Then, the distance D between P_1 and the plane is given by the length of the vector projection of $\vec{P_0P_1}$ onto the normal vector **n** $=\langle a, b, c \rangle$ of the plane. Equivalently, D is the absolute value of the scalar component of $\vec{P_0P_1}$ along **n**. Thus, we obtain

$$D = \frac{|\vec{P_0P_1}.\mathbf{n}|}{|\mathbf{n}|}$$



But $\vec{P_0P_1} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0\rangle$. Substituting into above equation and simplifying, we get

$$D = \frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

4.2 Surfaces in space

In the previous section we saw that the graph of a linear equation in three variables is a plane in space. In general, the graph of an equation in three variables, $F(x, y, z) = 0$, is a surface in 3-space. In this section we will study surfaces called cylinders and quadric surfaces.

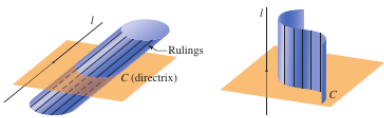
Traces

The trace of a surface S in a plane is the intersection of the surface and the plane. In particular, the traces of S in the xy-plane, the

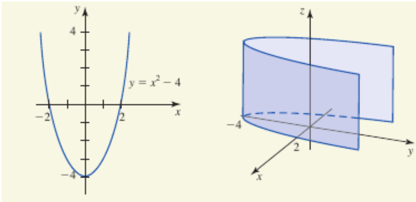
yz-plane, and the xz-plane are called the xy-trace, the yz-trace and the xz-trace, respectively. To find the xy-traces, we set $z = 0$ and sketch the graph of the resulting equation in the xy-plane. The other traces are obtained in a similar manner. Of course, if the surface does not intersect the plane, there is no trace in that plane.

Cylinders

Definition 32. *(**Cylinder**) Let C be a curve in a plane, and let l be a line that is not parallel to that plane. Then the set of all points generated by letting a line traverse C while parallel to l at all times is called a cylinder. The curve C is called the **directrix** of the cylinder, and each line through C parallel to l is called a ruling of the cylinder.*



Example : Sketch the graph of $y = x^2 - 4$.
Solution : The given equation has the form $f(x, y) = 0$, where $f(x, y) = x^2 - y - 4$. Therefore, its graph is a cylinder with directrix given by the graph of $y = x^2 - 4$ in the xy-plane and rulings parallel to the z-axis (corresponding to the variable missing in the equation). The graph of $y = x^2 - 4$ in the xy-plane is the parabola shown in Figure (a) below. The required cylinder is shown in Figure (b). It is called a parabolic cylinder.



(a) The directrix shown in the xy-plane (b) The parabolic cylinder $y = x^2 - 4$

Quadric surfaces

The graph of the second degree equation

$$Ax^x + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

where $A, B, C, ..., J$ are constants, is called a quadric surface.

Ellipsoid : The graph of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is an ellipsoid because its traces in the planes parallel to the coordinate planes are ellipses. In fact, its trace in the plane $z = k$, where $-c < k < c$, is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2}$$

and, in particular, its trace in the xy-plane is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

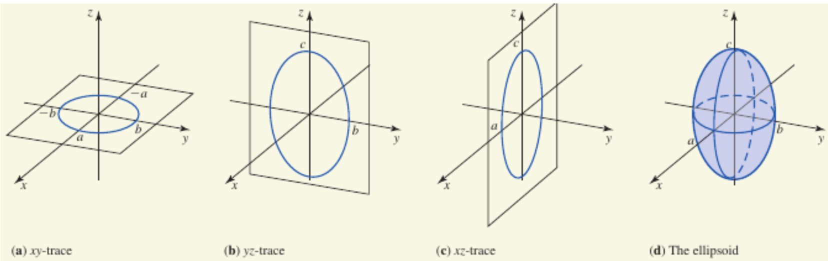
Similarly, its traces in the planes $x = k\ (-a < k < a)$ and $y = k(-b < k < b)$ are ellipses and, in particular, that its yz- and xz-traces are the ellipses

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

and

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$$

respectively.



Hyperboloid of One sheet : The graph of the equation

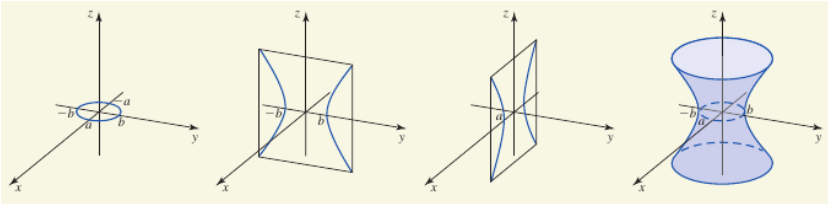
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

is a hyperboloid of one sheet. The xy-trace of this surface is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

whereas both the yz- and xz-traces are hyperbolas. The trace of the surface in the plane $z = k$ is an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{k^2}{c^2}$$



(a) xy-trace (b) yz-trace (c) xz-trace (d) A hyperboloid of one sheet

As $|k|$ increases, the ellipses grow larger and large. The $z-$axis is called the axis of the hyperboloid. Note that the orientation of the axis of the hyperboloid is associated with the term that has a minus sign in front of it. Thus, if the minus sign had been in front of the term involving x, then the surface would have been a hyperboloid of one sheet with the x-axis as its axis.

Hyperboloid of Two sheets : The graph of the equation

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is a hyperboloid of two sheets. The xz- and yz-traces are the hyperbolas

$$-\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$$

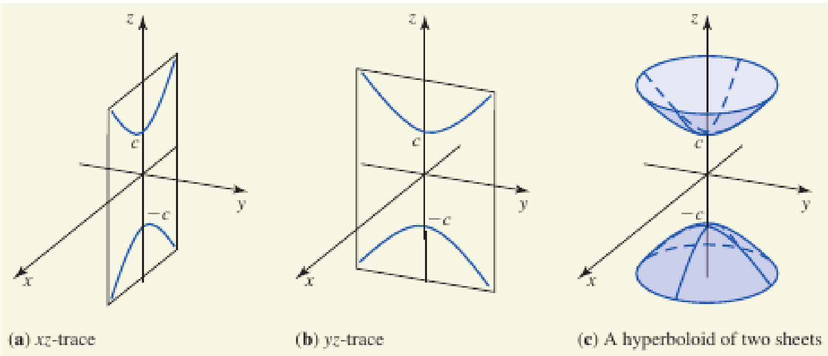
and

$$-\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

The trace of the surface in the plane $z = k$ is an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{k^2}{c^2} - 1$$

provided that $|k| > c$. There are no values of x and y that satisfy the equation if $|k| < c$, so the surface is made up of two parts: one part lying on or above the plane $z = c$ and the other part lying on or below the plane $z = -c$. The axis of the hyperboloid is the z-axis. Observe that the sign associated with the variable z is positive. Had the positive sign been in front of one of the other variables, then the surface would have been a hyperboloid of two sheets with its axis along the axis associated with that variable.



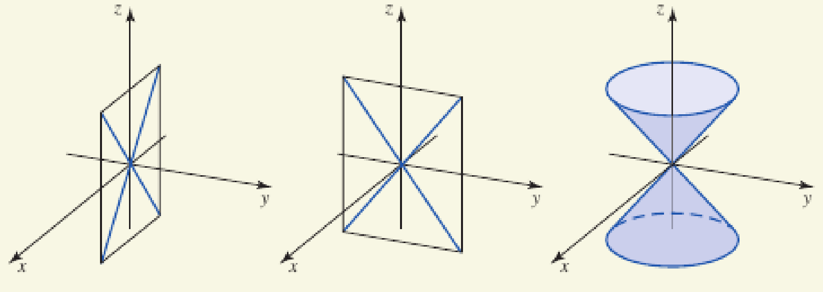
Cones : The graph of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

is a double-napped cone. The xz- and yz-traces are the lines $z = \pm(c/a)x$ and $z = \pm(c/b)y$, respectively. The trace in the plane $z = k$ is an ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{k^2}{c^2}$$

As $|k|$ increases, so do the lengths of the axes of the resulting ellipses. The traces in planes parallel to the other two coordinate planes are hyperbolas.

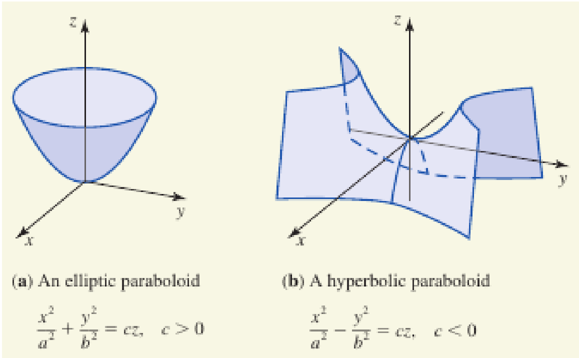


(a) xz-trace (b) yz-trace (c) A cone

Paraboloids : The graph of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = cz$$

where c is a real number, is called an elliptic paraboloid because its traces in planes parallel to the xy-coordinate plane are ellipses and its traces in planes parallel to the other two coordinate planes are parabolas. If $a = b$, the surface is called a circular paraboloid. The graph of an elliptic paraboloid with $c > 0$ is sketched in Figure (a) below. The axis of the paraboloid is the z-axis, and its vertex is the origin.



(a) An elliptic paraboloid

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = cz, \quad c > 0$

(b) A hyperbolic paraboloid

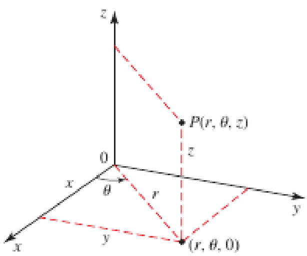
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = cz, \quad c < 0$

Hyperbolic paraboloid : The graph of the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = cz$$

where c is a real number, is called a hyperbolic paraboloid because the xz- and yz-traces are parabolas and the traces in planes parallel to the xy-plane are hyperbolas. The graph of a hyperbolic paraboloid with $c < 0$ is shown in Figure (b) above.

4.3 Cylindrical and spherical coordinates



The cylindrical coordinate system is just an extension of the polar coordinate system in the plane to a three-dimensional system in space obtained by adding the (perpendicular) z-axis to the system. A point P in this system is represented by the ordered triple (r, θ, z), where r and θ are the polar coordinates of the projection of P onto the xy-plane and z is the directed distance from $(r, \theta, 0)$ to P.

The relationship between rectangular coordinates and cylindrical coordinates can be seen by examining Figure. If P has representation (x, y, z) in terms of rectangular coordinates, then we have the following equations for converting cylindrical coordinates to rectangular coordinates and vice versa.

Converting cylindrical to rectangular coordinates:

$$x = rcos\theta \quad y = rsin\theta \quad z = z$$

Converting rectangular to cylindrical coordinates:

$$r^2 = x^2 + y^2 \quad tan\theta = \frac{y}{x} \quad z = z$$

Example : The point $(3, \pi/4, 3)$ is expressed in cylindrical coordinates. Find its rectangular coordinates.

Solution : We are given that $r = 3, \theta = \pi/4$, and $z = 3$. Using the above equations, we have

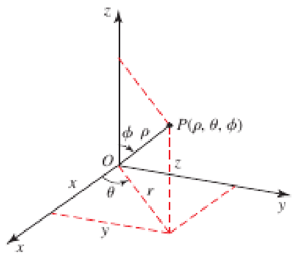
$$x = rcos\theta = 3cos\frac{\pi}{4} = \frac{3\sqrt{2}}{2}$$

$$x = rsin\theta = 3sin\frac{\pi}{4} = \frac{3\sqrt{2}}{2}$$

$$z = 3$$

Therefore, the rectangular coordinates of the given point are $(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}, 3)$.

The spherical coordinate system



In the spherical coordinate system a point P is represented by an ordered triple (ρ, θ, ϕ), where ρ is the distance between P and the origin, θ is the same angle as the one used in the cylindrical coordinate system, and ϕ is the angle between the positive z-axis and the line segment OP. Note that the spherical coordinates satisfy $\rho \geq 0$, $0 \leq \theta < 2\pi$, and $0 \leq \phi \leq \pi$.

The relationship between rectangular coordinates and spherical coordinates can be seen by examining the figure above. If P has representation (x, y, z)in terms of rectangular coordinates, then

$x = rcos\theta$ and $y = rsin\theta$

Since $r = \rho sin\phi$ and $z = \rho cos\phi$ we have the following equations for converting spherical coordinates to rectangular coordinates and vice versa.

Converting spherical to rectangular coordinates

$$x = \rho sin\phi cos\theta \quad y = \rho sin\phi sin\theta \quad z = \rho cos\phi$$

Converting rectangular to spherical coordinates

$$\rho^2 = x^2 + y^2 + z^2 \quad tan\theta = \frac{y}{x} \quad cos\phi = \frac{z}{\rho}$$

Example : Find an equation in spherical coordinates for the paraboloid with rectangular equation $4z = x^2 + y^2$.

Solution : Using the above equations, we obtain

$$\begin{aligned} 4\rho cos\phi &= \rho^2 sin^2\phi cos^2\theta + \rho^2 sin^2\phi sin^2\theta \\ &= \rho^2 sin^2\phi(cos^2\theta + sin^2\theta) \\ &= \rho^2 sin^2\phi \end{aligned}$$

or

$$\rho sin^2\phi = 4cos\phi$$

4.4 Vector-valued functions and space curves

Definition 33. *(**Vector function**) A **vector-valued function**, or **vector function**, is a function **r** defined by*

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

*where the component functions f, g, and h of **r** are real-valued functions of the parameter t lying in a **parameter interval I**.*

Example : Find the domain (parameter interval) of the vector function

$$\mathbf{r}(t) = \langle \frac{1}{t}, \sqrt{t-1}, lnt \rangle$$

The component functions of **r** are $f(t) = 1/t$, $g(t) = \sqrt{t-1}$, and $h(t) = lnt$. Observe that f is defined for all values of t except $t = 0$, g is defined for all $t \geq 1$, and h is defined for all $t > 0$. Therefore; f, g, and h are all defined if $t \geq 1$, and we conclude that the domain of **r** is $[1, \infty)$.

Curves defined by vector functions

A plane or space curve is the curve traced out by the terminal point of $\mathbf{r}(t)$ of a vector function **r** as t takes on all values in a parameter interval.

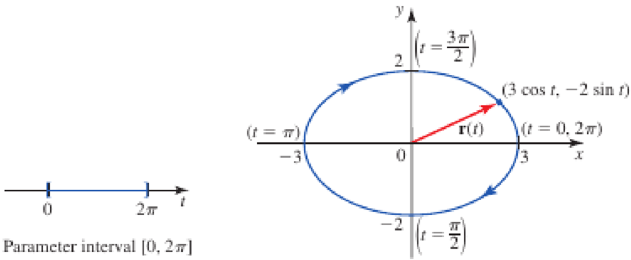
Example : Sketch the curve defined by the vector function

$$\mathbf{r}(t) = \langle 3cost, -2sint \rangle \quad 0 \leq t \leq 2\pi$$

Solution : The parametric equations for the curve are
$x = 3cost$ and $y = -2sint$ Solving the first equation for $cost$ and the second equation for $sint$ and using the identity $cos^2t + sin^2t = 1$, we obtain the rectangular equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

The curve described by this equation is the ellipse shown in Figure below. As t increases from 0 to 2π, the terminal point of **r** traces the ellipse in a clockwise direction.



Example : Sketch the curve defined by the vector function

$$\mathbf{r}(t) = 2cost\mathbf{i} + 2sint\mathbf{j} + t\mathbf{k} \quad 0 \leq t \leq 2\pi$$

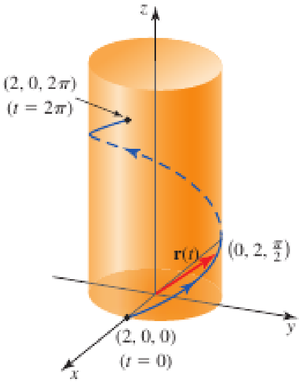
Solution : The parametric equations for the curve are

$$x = 2cost \quad y = 2sint \quad z = t$$

From the first two equations we obtain

$$(\frac{x}{2})^2 + (\frac{y}{2})^2 = cos^2t + sin^2t = 1 \quad or \quad x^2 + y^2 = 4$$

This says that the curve lies on the right circular cylinder of radius 2, whose axis is the z-axis. At $t = 0$, **r**(0) = 2**i**, and this gives $(2, 0, 0)$ as the starting point of the curve. Since $z = t$, the z-coordinate of the point on the curve increases (linearly) as t increases, and the curve spirals upward around the cylinder in a counterclockwise direction, terminating at the point $(2, 0, 2\pi)$ $[\mathbf{r}(2\pi) = 2\mathbf{i} + 2\pi\mathbf{k}]$. The curve, called a helix.



Limits and continuity

Definition 34. *(The limit of a Vector function) Let* $\mathbf{r}$ *be a function defined by* $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$. *Then*

$$\lim_{t\to a} \mathbf{r}(t) = [\lim_{t\to a} f(t)]\mathbf{i} + [\lim_{t\to a} g(t)]\mathbf{j} + [\lim_{t\to a} h(t)]\mathbf{k}$$

provided that the limits of the component functions exist.

Example : Find $\lim_{t\to 0} \mathbf{r}(t)$, where $\mathbf{r}(t) = \sqrt{t+2}\mathbf{i} + tcos2t\mathbf{j} + e^{-t}\mathbf{k}$.

Solution :

$$\begin{aligned}\lim_{t\to 0} &= [\lim_{t\to 0} \sqrt{t+2}]\mathbf{i} + [\lim_{t\to 0} icos2t]\mathbf{j} + [\lim_{t\to 0} e^{-t}]\mathbf{k} \\ &= \sqrt{2}\mathbf{i} + \mathbf{k}\end{aligned}$$

Definition 35. *(**Continuity of a vector function**) A vector function* $\mathbf{r}$ *is continuous at* a *if*

$$\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a)$$

*A vector function **r** is continuous on an interval I if it is continuous at every number in I.*

Example : Find the interval(s) on which the vector function **r** defined by

$$\mathbf{r}(t) = \sqrt{t}\mathbf{i} + (\frac{1}{t^2 - 1})\mathbf{j} + lnt\mathbf{k}$$

is continuous.

Solution : The component functions of **r** are $f(t) = \sqrt{t}$, $g(t) = 1/(t^2 - 1)$, and $h(t) = lnt$. Observe that f is continuous for $t \geq 0$, g is continuous for all values of t except $t = \pm 1$, and h is continuous for $t > 0$. Therefore, **r** is continuous on the intervals $(0, 1)$ and $(1, \infty)$.

4.5 Differentiation and integration of vector-valued functions

The derivative of a vector function

Definition 36. ***(Derivative of a vector function)*** *The derivative of a* ***vector function r*** *is the vector function* ***r'*** *defined by*

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{h \to 0} \frac{\mathbf{r}(t + h) - \mathbf{r}(t)}{h}$$

provided that the limit exists.

The derivative **r'** of the vector **r** at P may be interpreted as the tangent vector to the curve defined by **r** at the point P, provided that **r'**$(t) \neq 0$. If we divide **r'**(t) by its length, we obtain the unit tangent vector

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|r'(t)|}$$

which has unit length and the direction of **r'**.

Theorem 45. *(Differentiation of Vector functions) Let* $\mathbf{r}(t) = f(t)\mathbf{i}+g(t)\mathbf{j}+h(t)\mathbf{k}$*, where* f*,* g*, and* h *are differentiable functions of* t*. Then*

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

Proof. We compute

$\mathbf{r}'(t)$

$$\begin{aligned}
&= \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\delta t) - \mathbf{r}(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} [\frac{f(t+\Delta t)\mathbf{i} + g(t+\Delta t)\mathbf{j} + h(t+\Delta t)\mathbf{k}}{\Delta t} \\
&\qquad - \frac{[f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}]}{\Delta t}] \\
&= \lim_{\Delta t \to 0} [\frac{f(t+\Delta t) - f(t)}{\Delta t}\mathbf{i} + \frac{g(t+\Delta t) - g(t)}{\Delta t}\mathbf{j} + \frac{h(t+\Delta t) - h(t)}{\Delta t}\mathbf{k}] \\
&= [\lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}]\mathbf{i} + [\lim_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{\Delta t}]\mathbf{j} \\
&+ [\lim_{\Delta t \to 0} \frac{h(t+\Delta t) - h(t)}{\Delta t}]\mathbf{k} \\
&= f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}
\end{aligned}$$

$\square$

Example : a. Find the derivative of $\mathbf{r}(t) = (t^2+1)\mathbf{i} + e^{-t}\mathbf{j} - sin2t\mathbf{k}$.

b. Find the point of tangency and the unit tangent vector at the point on the curve corresponding to $t = 0$.

Solution : Using the theorem above, we get

$$\mathbf{r}'(t) = 2t\mathbf{i} - e^{-t}\mathbf{j} - 2cos2t\mathbf{k}$$

b. Since $\mathbf{r}(0) = \mathbf{i}+\mathbf{j}$, we see that the point of tangency is $(1, 1, 0)$. Next, since $\mathbf{r}'(0) = -\mathbf{j} - 2\mathbf{k}$, we find the unit tangent vector at $(1, 1, 0)$ to be

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{-\mathbf{j} - 2\mathbf{k}}{\sqrt{1+4}} = -\frac{1}{\sqrt{5}}\mathbf{j} - \frac{2}{\sqrt{5}}\mathbf{k}$$

Example : Find parametric equations for the tangent line to the helix with parametric equations

$$x = 3cost \quad y = 2sint \quad z = t$$

at the point where $t = \pi/6$.

Solution : The vector function that describes the helix is

$$\mathbf{r}(t) = 3cost\mathbf{i} + 2sint\mathbf{j} + t\mathbf{k}$$

The tangent vector at any point on the helix is

$$\mathbf{r}'(t) = -3sint\mathbf{i} + 2cost\mathbf{j} + \mathbf{k}$$

In particular, the tangent vector at the point $(\frac{3\sqrt{3}}{2}, 1, \frac{\pi}{6})$, where $t = \pi/6$, is

$$\mathbf{r}'(\frac{\pi}{6}) = -\frac{3}{2}\mathbf{i} + \sqrt{3}\mathbf{j} + \mathbf{k}$$

Finally, we the observe that the required tangent line passes through the point $(\frac{3\sqrt{3}}{2}, 1, \frac{\pi}{6})$ and has the same direction that the as required the tangent vector line $\mathbf{r}'(\pi/6)$. Thus, the parametric equations of this line are

$$x = \frac{3\sqrt{3}}{2} - \frac{3}{2}t, \quad y = 1 + \sqrt{3}t, \quad and \;\; z = \frac{\pi}{6} + t$$

Higher order derivatives

Higher-order derivatives of vector functions are obtained by successive differentiation of the lower-order derivatives of the function. For example, the second derivative of $\mathbf{r}(t)$ is

$$\mathbf{r}''(t) = \frac{d}{dt}\mathbf{r}'(t) = f''(t)\mathbf{i} + g''(t)\mathbf{j} + h''(t)\mathbf{k}$$

Rules of differentiation

Theorem 46. ***(Rules of differentiation)*** *Suppose that* **u** *and* **v** *are differentiable vector functions, f is a differentiable real-valued function, and c is a scalar. Then*

1. $\frac{d}{dt}[\mathbf{u}(t) \pm \mathbf{v}(t)] = \mathbf{u}'(t) \pm \mathbf{v}'(t)$

2. $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

3. $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$

4. $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$

5. $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$

6. $\frac{d}{dt}[\mathbf{u}(f(t))] = \mathbf{u}'(f(t))f'(t)$ Chain Rule

Example : Suppose that **v** is a differentiable vector function of

constant length c. Show that $\mathbf{v}.\mathbf{v}' = 0$. In other words, the vector **v** and its tangent vector **v'** must be orthogonal.
Solution : The condition on **v** implies that

$$\mathbf{v}.\mathbf{v} = |\mathbf{v}|^2 = c^2$$

Differentiating both sides of this equation with respect to t, we obtain

$$\frac{d}{dt}(\mathbf{v}.\mathbf{v}) = \mathbf{v}.\mathbf{v}' + \mathbf{v}'.\mathbf{v} = \frac{d}{dt}(c^2) = 0$$

But $\mathbf{v}'.\mathbf{v} = \mathbf{v}.\mathbf{v}'$, so we have

$$2\mathbf{v}.\mathbf{v}' = 0 \quad or \quad \mathbf{v}.\mathbf{v}' = 0$$

Integration of vector functions

Theorem 47. ***(Integration of vector functions)*** *Let* $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$*, where* f, g *and* h *are integrable. Then*
1. The ***indefinite integral of r with respect to t*** *is*

$$\int \mathbf{r}(t)dt = [\int f(t)dt]\mathbf{i} + [\int g(t)dt]\mathbf{j} + [\int h(t)dt]\mathbf{k}$$

2. The ***definite integral of r*** *over the interval* $[a, b]$ *is*

$$\int_a^b \mathbf{r}(t)dt = [\int_a^b f(t)dt]\mathbf{i} + [\int_a^b g(t)dt]\mathbf{j} + [\int_a^b h(t)dt]\mathbf{k}$$

Example : Find the antiderivative of $\mathbf{r}'(t) = cost\mathbf{i} + e^{-t}\mathbf{j} + \sqrt{t}\mathbf{k}$ satisfying the initial condition $\mathbf{r}(0) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

Solution : We have

$$\begin{aligned} \mathbf{r}(t) &= \int \mathbf{r}'(t)dt \\ &= \int (cost\mathbf{i} + e^{-t}\mathbf{j} + t^{1/2}\mathbf{k})dt \\ &= sint\mathbf{i} - e^{-t}\mathbf{j} + \frac{2}{3}t^{3/2}\mathbf{k} + C \end{aligned}$$

where C is a constant (vector) of integration. To determine C, we use the condition $\mathbf{r}(0) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ to obtain

$$\mathbf{r}(0) = C = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$$

Therefore,

$$\mathbf{r}(t) = (1 + sint)\mathbf{i} + (3 - e^{-t})\mathbf{j} + (3 + \frac{2}{3}t^{3/2})\mathbf{k}$$

4.6 Arc length and curvature

Arc length

We saw that the length of the plane curve given by the parametric equations $x = f(t)$ and $y = g(t)$, where $a \leq t \leq b$, is

$$L = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}dt$$

Now, suppose that C is described by the vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ instead. Then

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j}$$

and

$$|\mathbf{r}'(t)| = \sqrt{[f'(t)]^2 + [g'(t)]^2}$$

from which we see that L can also be written in the form

$$L = \int_a^{b|\mathbf{r}'|}(t)dt$$

Theorem 48. ***(Arc length of a space curve)*** *Let C be a curve given by the vector function*

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \quad a \leq t \leq b$$

where f', g', and h' are continuous. If C is traversed exactly once as t increases from a to b, then its length is given by

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}dt = \int_a^b |\mathbf{r}'(t)|dt$$

Example : Find the length of the arc of the helix C given by the vector function $\mathbf{r}(t) = 2cost\mathbf{i} + 2sint\mathbf{j} + t\mathbf{k}$, where $0 \leq t \leq 2\pi$.
Solution : We first compute

$$\mathbf{r}'(t) = -2sint\mathbf{i} + 2cost\mathbf{j} + \mathbf{k}$$

Then, the length of the arc is

$$L = \int_0^{2\pi} |\mathbf{r}'(t)|dt = \int_0^{2\pi} \sqrt{4sin^2t + 4cos^2t + 1}dt$$
$$= \int_0^{2\pi} \sqrt{5}dt = 2\sqrt{5}\pi$$

Smooth curve

A curve that is defined by a vector function $\mathbf{r}$ on a parameter interval I is said to be smooth if $\mathbf{r}'(t)$ is continuous and $\mathbf{r}'(t) \neq 0$ for all t in I with the possible exception of the endpoints. For example, the plane curve defined by $\mathbf{r}(t) = t^3\mathbf{i} + t^2\mathbf{j}$ is smooth everywhere except at the point $(0, 0)$ corresponding to $t = 0$. To see this, we compute $\mathbf{r}'(t) = 3t^2\mathbf{i} + 2t\mathbf{j}$ and note that $\mathbf{r}'(0) = 0$. The point $(0, 0)$ where the curve has a sharp corner is called a cusp.

Arc length parameter

The curve C described by the vector function $\mathbf{r}(t)$ with parameter t in some parameter interval I is said to be parametrized by t. A curve C can have more than one parametrization. For example, the helix represented by the vector function

$$\mathbf{r}_1(t) = 2cost\mathbf{i} + 3sint\mathbf{j} + t\mathbf{k} \quad 2\pi \leq t \leq 4\pi$$

with parameter t is also represented by the function

$$\mathbf{r}_2(t) = 2cose^u\mathbf{i} + 3sine^u\mathbf{j} + e^u\mathbf{k} \quad ln2\pi \leq u \leq ln4\pi$$

with parameter u, where t and u are related by $t = e^u$.

A useful parametrization of a curve C is obtained by using the arc length of C as its parameter. To see how this is done, we need the following definition.

Definition 37. *(**Arc length function**) Suppose that C is a smooth curve described by $\mathbf{r}(t) = f(t)\mathbf{i}+g(t)\mathbf{j}+h(t)\mathbf{k}$, where $a \leq t \leq b$. Then the arc length function s is defined by*

$$s(t) = \int_a^t |\mathbf{r}'(u)|du$$

Differentiating $s(t)$ with respect to t and using the Fundamental theorem of calculus, we obtain

$$\frac{ds}{dt} = |\mathbf{r}'(t)|$$

or, in differential form,

$$ds = |\mathbf{r}'(t)|dt$$

The following example shows how to parametrize a curve in terms of its arc length.

Example : Find the arc length function $s(t)$ for the circle C in the plane described by

$$\mathbf{r}(t) = 2cost\mathbf{i} + 2sint\mathbf{j} \quad 0 \leq t \leq 2\pi$$

Then use the result to find a parametrization of C in terms of s.
Solution : We first compute $\mathbf{r}'(t) = -2sint\mathbf{i} + 2cost\mathbf{j}$, and then

compute

$$|\mathbf{r}'(t)| = \sqrt{4sin^2t + 4cos^2t} = 2$$

Then,

$$s(t) = \int_0^t 2du = 2t \quad 0 \leq t \leq 2\pi$$

Writing s for $s(t)$, we have $s = 2t$, where $0 \leq t \leq 2\pi$, which when solved for t, yields $t = t(s) = s/2$. Substituting this value of t into the equation for $\mathbf{r}(t)$ gives

$$\mathbf{r}(t(s)) = 2cos(\frac{s}{2})\mathbf{i} + 2sin(\frac{s}{2})\mathbf{j}$$

Finally, since $s(0) = 0$ and $s(2\pi) = 4\pi$, we see that the parameter interval for this parametrization by the arc length s is $[0, 4\pi]$.

Note : One reason for using the arc length of a curve C as the parameter stems from the fact that its tangent vector $\mathbf{r}'(s)$ has unit length; that is, $\mathbf{r}'(s)$ is a unit tangent vector. Consider the circle of example above. Here,

$$\mathbf{r}'(s) = -sin(\frac{s}{2})\mathbf{i} + cos(\frac{s}{2})\mathbf{j}$$

so

$$|\mathbf{r}'(s)| = \sqrt{sin^2(\frac{s}{2}) + cos^2(\frac{s}{2})} = 1$$

Curvature

Definition 38. *(**Curvature**) Let C be a smooth curve defined by* $\mathbf{r}(s)$ *, where s is the arc length of the parameter. Then the curva-*

ture of C at s is

$$\kappa(s) = |\frac{d\mathbf{T}}{ds}| = |\mathbf{T}'(s)|$$

where $\mathbf{T}$ is the unit tangent vector.

Although the use of the arc length parameter s provides us with a natural way for defining the curvature of a curve, it is generally easier to find the curvature in terms of the parameter t. Applying chain rule $\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt}$ Then

$$\kappa(s) = |\frac{d\mathbf{T}}{ds}| = \frac{|\frac{d\mathbf{T}}{dt}|}{|\frac{ds}{dt}|}$$

Since $ds/dt = |\mathbf{r}'(t)|$, we are led to the following formula:

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

Theorem 49. ***(Formula for finding curvature)*** *Let C be a smooth curve given by the vector function $\mathbf{r}$. Then the curvature of C at any point on C corresponding to t is given by*

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

Proof. We begin by recalling that

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

Since $|\mathbf{r}'(t)| = ds/dt$, we have

$$\mathbf{r}'(t) = \frac{ds}{dt}\mathbf{T}(t)$$

Differentiating both sides of this equation with respect to t

$$\mathbf{r}''(t) = \frac{d^2s}{dt^2}\mathbf{T}(t) + \frac{ds}{dt}\mathbf{T}'(t)$$

Next, we use the fact that $\mathbf{T} \times \mathbf{T} = 0$ to obtain

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = (\frac{ds}{dt})^2(\mathbf{T}(t) \times \mathbf{T}'(t))$$

Also, $|\mathbf{T}(t)| = 1$ for all t implies that $\mathbf{T}(t)$ and $\mathbf{T}'(t)$ are orthogonal.(as seen in the example in the previous section). Therefore, we have

$$|\mathbf{r}'(t)\times\mathbf{r}''(t)| = (\frac{ds}{dt})^2|\mathbf{T}(t)\times\mathbf{T}'(t)| = (\frac{ds}{dt})^2|\mathbf{T}(t)||\mathbf{T}'(t)| = (\frac{ds}{dt})^2|\mathbf{T}'(\mathbf{t})|$$

Upon solving for $|\mathbf{T}'(t)|$, we obtain

$$|\mathbf{T}'(t)| = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{(\frac{ds}{dt})^2} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^2}$$

from which we deduce that

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{\mathbf{r}'(t)} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

$\square$

Theorem 50. ***(Formula for the curvature of the graph of a function)*** *If C is the graph of a twice differentiable function f, then the curvature at the point (x, y) where $y = f(x)$ is given by*

$$\kappa(x) = \frac{|f''(x)|}{[1 + [f'(x)]^2]^{3/2}} = \frac{|y''|}{[1 + (y')^2]^{3/2}}$$

Proof. Using x as the parameter, we can represent C by the vector function $\mathbf{r}(x) = x\mathbf{i}+f(x)\mathbf{j}+0\mathbf{k}$. Differentiating $\mathbf{r}(x)$ with respect to x successively, we obtain

$$\mathbf{r}'(x) = \mathbf{i} + f'(x)\mathbf{j} + 0\mathbf{k} \quad and \quad \mathbf{r}''(x) = 0\mathbf{i} + f''(x)\mathbf{j} + 0\mathbf{k}$$

from which we obtain

$$\mathbf{r}'(x) \times \mathbf{r}''(x) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(x) & 0 \\ 0 & f''(x) & 0 \end{vmatrix} = f''(x)\mathbf{k}$$

and

$$|\mathbf{r}'(t) \times \mathbf{r}''(x)| = |f''(x)|$$

Also,

$$|\mathbf{r}'(x)| = \sqrt{1 + [f'(x)]^2}$$

Therefore,

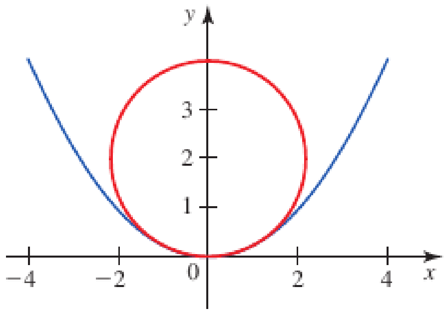
$$\kappa(x) = \frac{|\mathbf{r}'(x) \times \mathbf{r}''(x)|}{|\mathbf{r}'(x)|^3} = \frac{|f''(x)|}{[1 + [f'(x)]^2]^{3/2}}$$

□

Radius of curvature

Suppose that C is a plane curve with curvature κ at the point P. Then the reciprocal of the curvature, $\rho = 1/\kappa$, is called the **radius of curvature** of C at P. The radius of curvature at any point P on a curve C is the radius of the circle that best "fits" the curve at that point. This circle, which lies on the concave side of the curve and shares a common tangent line with the curve at P, is called the **circle of curvature** or **osculating circle**.

The center of the circle is called the **center of curvature**. As an example, the curvature of the parabola $y = 1/4x^2$ at the point $(0, 0)$ is found to be $1/2$. Therefore, the radius of curvature of the parabola at $(0, 0)$ is $\rho = 2$. The circle of curvature is shown in the figure below. Its equation is $x^2 + (y - 2)^2 = 4$.



4.7 Velocity and acceleration

Velocity, acceleration, and speed

Definition 39. *(**Velocity, acceleration, and speed**) Let* $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ *be the position vector of an object. If* $f, g,$ *and* h *are twice differentiable functions of* t*, then the **velocity vector*** $\mathbf{v}(t)$*, **acceleration vector*** $\mathbf{a}(t)$*, and **speed*** $|\mathbf{v}(t)|$ *of the object at time* t *are defined by*

$$\mathbf{v}(t) = \mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = f''(t)\mathbf{i} + g''(t)\mathbf{j} + h''(t)\mathbf{k}$$

$$|\mathbf{v}(t)| = |\mathbf{r}'(t)| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$$

Example : The position of an object moving in a plane is given by

$$\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} \quad t \geq 0$$

Find its velocity, acceleration, and speed when $t = 2$.

Solution : The velocity and acceleration vectors of the object are

$$\mathbf{v}(t) = \mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$$

and

$$\mathbf{a}(t) = \mathbf{r}''(t) = 2\mathbf{i}$$

Therefore, its velocity, acceleration, and speed when $t = 2$ are

$$\mathbf{v}(2) = 4\mathbf{i} + \mathbf{j}$$

$$\mathbf{a}(2) = 2\mathbf{i}$$

and

$$|\mathbf{v}(2)| = \sqrt{16 + 1} = \sqrt{17}.$$

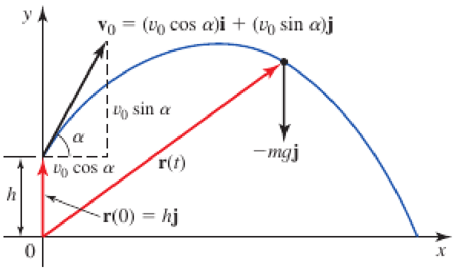
Motion of a particle

A projectile of mass m is fired from a height h with an initial velocity $\mathbf{v}_0$ and an angle of elevation α. If we describe the position of the projectile at any time t by the position vector $\mathbf{r}(t)$, then its initial position may be described by the vector

$$\mathbf{r}(0) = h\mathbf{j}$$

and its initial velocity by the vector

$$\mathbf{v}(0) = \mathbf{v}_0 = (v_0 cos\alpha)\mathbf{i} + (v_0 sin\alpha)\mathbf{j} \quad v_0 = |\mathbf{v}_0|$$



If we assume that air resistance is negligible and that the only external force acting on the projectile is due to gravity, then the force acting on the projectile during its flight is

$$\mathbf{F} = -mg\mathbf{j}$$

where g is the acceleration due to gravity. By Newton's Second Law of Motion this force is equal to ma, where a is the acceleration of the projectile. Therefore,

$$m\mathbf{a} = -mg\mathbf{j}$$

giving the acceleration of the projectile as

$$\mathbf{a}(t) = -g\mathbf{j}$$

To find the velocity of the projectile at any time t, we integrate the last equation with respect to t to obtain

$$\mathbf{v}(t) = \int -g\mathbf{j}dt = -gt\mathbf{j} + C$$

Setting $t = 0$ and using the initial condition $\mathbf{v}(0) = \mathbf{v}_0$, we obtain

$$\mathbf{v}(0) = C = \mathbf{v}_0$$

Therefore, the velocity of the projectile at any time t is

$$\mathbf{v}(t) = -gt\mathbf{j} + \mathbf{v}_0$$

Integrating this equation then gives

$$\mathbf{r}(t) = \int (-gt\mathbf{j} + \mathbf{v}_0)dt = \frac{1}{2}gt^2\mathbf{j} + \mathbf{v}_0 t + D$$

Setting $t = 0$ and using the initial condition $\mathbf{r}(0) = h\mathbf{j}$, we obtain

$$\mathbf{r}(0) = D = h\mathbf{j}$$

Therefore, the position of the projectile at any time t is

$$\mathbf{r}(t) = -\frac{1}{2}gt^2\mathbf{j} + \mathbf{v}_0 t + h\mathbf{j}$$

or

$$\mathbf{r}(t) = -\frac{1}{2}gt^2\mathbf{j} + [(v_0 cos\alpha)\mathbf{i} + (v_0 sin\alpha)\mathbf{j}]t + h\mathbf{j}$$
$$= (v_0 cos\alpha)t\mathbf{i} + [h + (v_0 sin\alpha)t - \frac{1}{2}gt^2]\mathbf{j}$$

4.8 Tangential and normal components of acceleration

The unit normal

Suppose that C is a smooth space curve described by the vector function $\mathbf{r}(t)$. Then,

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \quad \mathbf{r}'(t) \neq 0$$

is the unit tangent vector to the curve C at the point corresponding to t. Since $|\mathbf{T}(t)| = 1$ for every t, then the vector $\mathbf{T}'(t)$ is orthogonal to $\mathbf{T}(t)$(as seen in an earlier example). Therefore, if $\mathbf{r}'$ is also smooth, we can normalize $\mathbf{T}'(t)$ to obtain a unit vector that is orthogonal to $\mathbf{T}(t)$. This vector

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

is called the **principal unit normal vector** (or simply the unit normal) to the curve C at the point corresponding to t.

Example : Let C be the helix defined by

$$\mathbf{r}(t) = 2cost\mathbf{i} + 2sint\mathbf{j} + t\mathbf{k} \quad t \geq 0$$

Find $\mathbf{T}(t)$ and $\mathbf{N}(t)$.
Solution : Since

$$\mathbf{r}'(t) = -2sint\mathbf{i} + 2cost\mathbf{j} + \mathbf{k}$$

and

$$|\mathbf{r}'(t)| = \sqrt{4sin^2t + 4cos^2t + 1} = \sqrt{5}$$

we have

$$\mathbf{T}(t) = \frac{1}{\sqrt{5}}(-2sint\mathbf{i} + 2cost\mathbf{j} + \mathbf{k})$$

Now, differentiating $\mathbf{T}$

$$\mathbf{T}'(t) = -\frac{2}{\sqrt{5}}(cost\mathbf{i} + sint\mathbf{j})$$

and

$$|\mathbf{T}'(t)| = \frac{2}{\sqrt{5}}$$

it follows that

$$\mathbf{N}(t) = -(cost\mathbf{i} + sint\mathbf{j})$$

Tangential and normal components of acceleration

Let's return to the study of the motion of an object moving along the curve C described by the vector function $\mathbf{r}$ defined on the parameter interval I. Recall that the speed v of the object at any time t is $v = |\mathbf{v}(t)| = |\mathbf{r}'(t)|$. But

$$\mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

so we can write

$$\mathbf{v}(t) = \mathbf{r}'(t) = |\mathbf{r}'(t)|\mathbf{T} = v\mathbf{T}$$

which expresses the velocity of the object in terms of its speed and direction.

The acceleration of the object at time t is

$$\mathbf{a} = \mathbf{v}' = \frac{d}{dt}(v\mathbf{T}) = v'\mathbf{T} + v\mathbf{T}'$$

To obtain an expression for $\mathbf{T}'$, recall that

$$\mathbf{N} = \frac{\mathbf{T}'}{|\mathbf{T}'|}$$

so $\mathbf{T}' = |\mathbf{T}'|\mathcal{N}$. Now we need an expression for $|\mathbf{T}'|$. But

$$\kappa = \frac{|\mathbf{T}'|}{|\mathbf{r}'|}$$

where κ is the curvature of C. This gives

$$|\mathbf{T}'| = \kappa|\mathbf{r}'| = \kappa v$$

so $\mathbf{T}' = |\mathbf{T}'|\mathbf{N} = \kappa v\mathbf{N}$.
Therefore,

$$\mathbf{a} = v'\mathbf{T} + \kappa v^2\mathbf{N})$$

This result shows that the acceleration vector $\mathbf{a}$ can be resolved into the sum of two vectors—one along the tangential direction and the other along the normal direction. The magnitude of the acceleration along the tangential direction is called the **tangential scalar component of acceleration** and is denoted by a_T, whereas the magnitude of the acceleration along the normal direction is called the **normal scalar component of acceleration** and is denoted by a_N. Thus,

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$$

where $a_T = v'$ and $a_N = \kappa v^2$.

The following theorem gives formulas for calculating a_T and a_N directly from $\mathbf{r}$ and its derivatives.

Theorem 51. ***(Tangential and normal components of acceleration)*** *Let $\mathbf{r}(t)$ be the position vector of an object moving along a smooth curve C. Then*

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$$

where

$$a_T = \frac{\mathbf{r}'(t).\mathbf{r}''(t)}{|\mathbf{r}'(t)|}$$

and

$$a_N = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{|\mathbf{r}'(t)|}$$

Proof. If we take the dot product of $\mathbf{v}$ and $\mathbf{a}$, we obtain

$$\begin{aligned}\mathbf{v}.\mathbf{a} &= (v\mathbf{T}).(v'\mathbf{T} + \kappa v^2\mathbf{N}) \\ &= vv'\mathbf{T}.\mathbf{T} + \kappa v^3\mathbf{T}.\mathbf{N}\end{aligned}$$

But $\mathbf{T}.\mathbf{T} = |\mathbf{T}|^2 = 1$, since $\mathbf{T}$ is a vector, and $\mathbf{T}.\mathbf{N} = 0$, since $\mathbf{T}$ and $\mathbf{N}$ are orthogonal. Therefore,

$$\mathbf{v}.\mathbf{a} = vv'$$

or

$$a_T = v' = \frac{\mathbf{v}.\mathbf{a}}{v} = \frac{\mathbf{r}'(t).\mathbf{r}''(t)}{|\mathbf{r}'(t)|}$$

Now,

$$a_N = \kappa v^2 = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}|\mathbf{r}'(t)|^2 = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|}$$

$\square$

Example : A particle moves along a curve described by the vector function $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$. Find the tangential scalar and normal scalar components of acceleration of the particle at any time t.

Solution : We begin by computing

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$$

$$\mathbf{r}''(t) = 2\mathbf{j} + 6t\mathbf{k}$$

Then,

$$a_T = \frac{\mathbf{r}'(t).\mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \frac{4t + 18t^3}{\sqrt{1 + 4t^2 + 9t^4}}$$

Next, we compute

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = 6t^2\mathbf{i} - 6t\mathbf{j} + 2\mathbf{k}$$

Then, we have

$$a_T = \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \frac{\sqrt{36t^4 + 36t^2 + 4}}{\sqrt{1 + 4t^2 + 9t^4}} = 2\sqrt{\frac{9t^4 + 9t^2 + 1}{9t^4 + 4t^2 + 1}}$$